Lecture 4 (10/24)

Tuesday, October 24, 2023 7:53 AM

Recap: Stiefel-Whitney classes

ξ = vector bundle

Axiom 1 $w_i(\xi) \in H^i(B, \mathbb{Z}_2)$

$w_0(\xi) = 1 \qquad w_i(\xi) = 0$ for $i > \mathrm{rank}(\xi)$

Axiom 2 (naturality) $\eta = f^*\xi$

$$\begin{tikzcd} \eta = f^*\xi \arrow[r] \arrow[d] & \xi \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \underbrace{w_i(\eta)}_{\in H^i(X)} = f^* \underbrace{w_i(\xi)}_{\in H^i(Y)}$$

Axiom 3 (Whitney sum) $w(\xi \oplus \eta) = w(\xi)\, w(\eta)$

$$w(\xi) = w_0(\xi) + w_1(\xi) + \dots + w_{\mathrm{rank}(\xi)}(\xi)$$

Axiom 4 $\gamma_1^1 \longrightarrow \mathbb{P}^1$ tautological bundle

$$w_1(\gamma_1^1) \neq 0 \iff w_1(\gamma_1^1) = a$$

a = generator of $H^1(\mathbb{P}^1)$.

Computations | ε = trivial bundle $\quad w(\varepsilon) = 1$

(1) $S^n \hookrightarrow \mathbb{R}^{n+1} \qquad \underset{\varepsilon}{NS^n}$ is trivial! $\Rightarrow w(NS^n) = 1$

$$TS^n \oplus NS^n = \varepsilon^{n+1}$$

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Whitney sum: $w(TS^n) \cdot 1 = 1 \Rightarrow \boxed{w(TS^n) = 1}$

(2)

$\gamma_1^1 \longrightarrow \gamma_n^1 \qquad$ **Exercise** $\gamma_1^1 = f^*\gamma_n^1$

$$\begin{array}{ccc} \gamma_1^1 & \longrightarrow & \gamma_n \\ \downarrow & & \downarrow \\ \mathbb{R}P^1 & \xhookrightarrow{f} & \mathbb{R}P^n \end{array}$$

<u>Exercise</u> $\gamma_1^1 = f^* \gamma_n$

$H^*(\mathbb{R}P^n) = \mathbb{Z}_2[a]/_{a^{n+1}=0}$

$$\begin{array}{cccc} f^*: & H^*(\mathbb{R}P^n) & \longrightarrow & H^*(\mathbb{R}P^1) \\ & \wr\wr & & \| \\ & \mathbb{Z}_2[a]/_{a^{n+1}} & \twoheadrightarrow & \mathbb{Z}_2[a]/_{a^2} \end{array}$$

$\omega(\gamma_n^1) = 1 + ka$ (Axiom 1) $k \in \mathbb{Z}_2$

$\omega(\gamma_1^1) = f^*(1 + ka) = 1 + ka = 1 + a \Rightarrow k = 1$.

<u>So</u>: $\boxed{\omega(\gamma_n^1) = 1 + a}$

(3) $\gamma_n^1 \oplus \gamma_n^\perp = \varepsilon^{n+1}$

$\omega(\gamma_n^1) \cdot \omega(\gamma_n^\perp) = 1$

$$\boxed{\begin{aligned} \omega(\gamma_n^\perp) &= \frac{1}{1+a} = \\ &= 1 + a + \dots + a^n \end{aligned}}$$

$(\bmod 2,\ a^{n+1} = 0)$.

(4) <u>Last time</u>: $T\mathbb{R}P^n = \mathrm{Hom}(\gamma_n^1, \gamma_n^\perp)$

$$\mathrm{Hom}(\gamma_n^1, \gamma_n^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \simeq \mathrm{Hom}(\gamma_n^1, \mathbb{R}^{n+1}) =$$

$$= \underbrace{(\gamma_n^1)^* \oplus \dots \oplus (\gamma_n^1)^*}_{n+1}$$

<u>Now</u>:

- $\gamma_n^1 \simeq (\gamma_n^1)^*$ (use metric)
- $\omega(\underbrace{\gamma_n^1 \oplus \dots \oplus \gamma_n^1}_{n+1}) \underset{\text{Whitney sum}}{=} \omega(\gamma_n^1)^{n+1} = (1+a)^{n+1}$
- $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ is trivial! Rank 1 bundle

- $\mathrm{Hom}(\gamma_n', \gamma_n')$ is trivial! Rank 1, has a nonzero section I.

(More generally, $\mathrm{Hom}(\mathcal{L}, \mathcal{L})$ is trivial for a line bdle $\mathcal{L}$).

<u>Conclusion</u> $\boxed{w(T\mathbb{R}P^n) = (1+a)^{n+1}}$

Pascal triangle mod 2

$$\begin{array}{ccccccccccccccccccl}
&&&&&&&&1&&&&&&&&& \leftarrow 0\\
&&&&&&&1&&1&&&&&&&& \leftarrow 1\\
&&&&&&1&&0&&1&&&&&&& \leftarrow 2\\
&&&&&1&&1&&1&&1&&&&&& \leftarrow 3\\
&&&&1&&0&&0&&0&&1&&&&& \leftarrow 4\\
&&&1&&1&&0&&0&&1&&1&&&& \leftarrow 5\\
&&1&&0&&1&&0&&1&&0&&1&&& \leftarrow 6\\
&1&&1&&1&&1&&1&&1&&1&&1&& \leftarrow 7\\
1&&0&&0&&0&&0&&0&&0&&0&&1& \leftarrow 8
\end{array}$$

<u>Ex</u> $w(T\mathbb{R}P^6) = (1+a)^7 = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6$

$w(T\mathbb{R}P^7) = (1+a)^8 = 1$ (since $a^8 = 0$).

<u>Detour</u> $n+1 = 2^r \cdot m$, m odd $\qquad (x+y)^2 = x^2 + y^2$ mod 2

$$(1+a)^{n+1} = \left[(1+a)^m\right]^{2^r} = (1 + ma + \dots)^{2^r} =$$

$$= (1 + a + \dots)^{2^r} = 1 + a^{2^r} + \dots \qquad (*)$$

<u>Thm</u> $w(T\mathbb{R}P^n) = 1$ iff $n+1 = 2^r$

<u>Pf</u> If $n+1 = 2^r$, follows from $(*)$

If $n+1 = 2^r \cdot m$, $m > 1$, then $2^r < n+1$ and $a^{2^r} \neq 0$.

Cor TS^n trivial $\Rightarrow T\mathbb{R}P^n$ trivial $\Rightarrow \omega(T\mathbb{R}P^n) = 1$

$\Rightarrow n = 2^r - 1$.

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(5) Suppose we can embed $\mathbb{R}P^n \hookrightarrow \mathbb{R}^N$

$$T\mathbb{R}P^n \oplus N\mathbb{R}P^n \simeq \varepsilon^N$$

$$\omega(N\mathbb{R}P^n) = \frac{1}{\omega(T\mathbb{R}P^n)} = \frac{1}{(1+a)^{n+1}}$$

Ex $n = 2^r \qquad (1+a)^n = 1 + a^{2^r} = 1 + a^n$

$$(1+a)^{n+1} = 1 + a + a^n$$

$$\frac{1}{(1+a)^{n+1}} = \frac{1}{(1+a^n)} \cdot \frac{1}{(1+a)} = (1+a^n)(1+a+\dots+a^n) =$$

$$= 1 + a + \dots + a^{n-1}$$

$$\omega_{n-1}(N\mathbb{R}P^n) \neq 0 \Rightarrow \operatorname{rank}(N\mathbb{R}P^n) \geq n-1$$

$$\Rightarrow \boxed{N \geq 2n-1}$$

Stiefel-Whitney numbers

$\omega_i = \omega_i(TM)$

M = smooth, compact, dim $= m$

$$k_1 + \dots + k_s = m \qquad \langle \underbrace{\omega_{k_1}(TM) \cdots \omega_{k_s}(TM)}_{H^m(M;\mathbb{Z}_2)}, \underbrace{[M]}_{H_m(M;\mathbb{Z}_2)} \rangle$$

Thm (Pontrjagin) If $M = \partial N$ then all SW numbers are zero.

Proof $M = \partial N$, long exact sequences

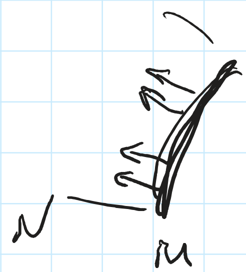
$$\to H_i(M) \to H_i(N) \to H_i(N,M) \xrightarrow{\partial} H_{i-1}(M)$$

$$\begin{tikzcd} \leftarrow H^i(M) & H^i(N) \arrow[l] & H^i(N,M) \arrow[l] & H^{i-1}(M) \arrow[l, "\delta"'] \\ & & & H^{i-1}(N) \arrow[u, "j^*"'] \end{tikzcd}$$

$[N] \in H_{n+1}(N,M)$ $\quad \partial[N] = [M]$

$$\langle \alpha, \underbrace{\partial[N]}_{[M]} \rangle = \langle \delta(\alpha), [N] \rangle$$

$\alpha \in H^n(M)$.



$$TN|_M = TM \oplus \varepsilon$$

(prove it!)

$w_i(TM) = j^* w_i(TN)$ $\qquad j : M \hookrightarrow N$

$\alpha = w_{k_1}(TM) \cdots w_{k_s}(TM) \in \operatorname{Im} j^* = \operatorname{Ker} \delta$

$\Rightarrow \langle \delta(\alpha), [N] \rangle = \langle \alpha, [M] \rangle = 0$. ∎