Chapter 5 (Yuze)

V – vector space $\dim n$

$U \subset V \quad \dim U = k \qquad Gr(k,n) = \{\text{all } k\text{-dim subspaces}\}$

Def Stiefel manifold is an open subset of $\mathbb{R}^n \times \dots \times \mathbb{R}^n = \{(v_1 \dots v_k) \mid v_1 \dots v_k \text{ lin. indep}\}$
$= S(k,n)$

Quotient map: $f: S(k,n) \longrightarrow Gr(k,n)$

$f: (v_1 \dots v_k) \longrightarrow \mathrm{Span}(v_1 \dots v_k)$

Def Topology on $Gr(k,n)$

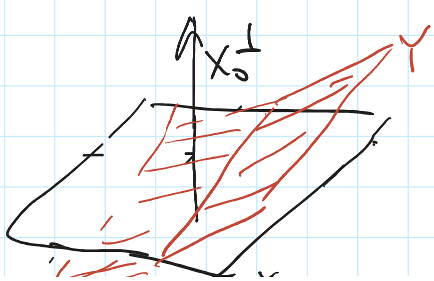
$U \subset Gr(k,n)$ is open if $f^{-1}(U)$ is open.

Ex $Gr(1,n) = \mathbb{R}P^{n-1}$

Prop $Gr(k,n)$ is a compact manifold of dimension $k(n-k)$.

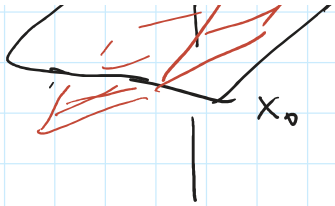
Proof Construct coordinate chart $U \subset Gr(k,n)$

$U \simeq \mathbb{R}^{k(n-k)}$



$x_0 \in Gr(k,n)$

$U = \{Y \subset \mathbb{R}^n, \dim Y = k, Y \cap X_0^\perp = \emptyset\}$



X_0

$\mathcal{U} = \{ Y \subset \mathbb{R}^n, \dim Y = k, Y \cap X_0^\perp = \emptyset \}$

$$\mathcal{U} \cong \mathrm{Hom}(\underbrace{X_0}_{k}, \underbrace{X_0^\perp}_{n-k}) \ni T$$

p_1 : projecting $x \in X_0$ along of $X_0^\perp$, $p_1 : X_0 \to Y$

p_2 : project onto $X_0^\perp$, $p_2 : Y \to X_0^\perp$

$T = p_2 \circ p_1$ linear transformation

$Y \longleftrightarrow T$ bijective, continuous

$\bar{x}_1 \dots \bar{x}_k$ = basis of X_0 $\qquad y_i = x_i + \underbrace{T(Y) x_i}_{\in X_0^\perp}$

$\updownarrow \qquad \updownarrow$

$y_1 \dots y_k$ = basis of Y

$\square$

<u>Def</u> The universal/tautological bundle on $Gr(k,n)$

$$\gamma_k = \{ (x, v) \mid x \in Gr(k,n), v \text{ in the } k\text{-plane corresponding to } x \}$$

<u>Prop</u> γ_k is locally trivial on $Gr(k,n)$

<u>Proof</u> $h : \mathcal{U} \times X_0 \longrightarrow \pi^{-1}(\mathcal{U})$

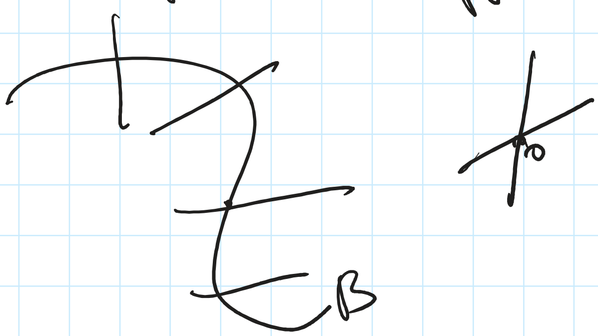
$h(x) = x + T(Y) x$ $\qquad$ Check: h is a homeomorphism.

Prop For any $\mathbb{R}^k$-bundle with compact base B

we can map it into $\gamma_k(Gr(k,n))$. for sufficiently large n.

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \gamma_k \arrow[d] \\ B \arrow[r, "f"] & Gr(k,n) \end{tikzcd}$$

Idea: Embed B into $\mathbb{R}^n$ for large n, and embed E so that all fibers are affine subspaces of $\mathbb{R}^n$



Transport these to the origin, we get a point in $Gr(k,n)$.

$\mathbb{R}^\infty = (x, \dots)$ all but finitely many are zero

direct limit $\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3 \subseteq \dots$

Def Infinite Grassmannian

$Gr_n = Gr(n, \infty) = \{n\text{-dim vector subspaces of } \mathbb{R}^\infty\}$

$=$ direct limit of $Gr(k,n) \subseteq Gr(k, k+1) \subseteq \dots$

Prop γ_k = the universal bundle over $Gr(k, \infty)$

Prop γ_n = the universal bundle over $Gr(k,\infty)$ is well defined and locally trivial.

Thm (5.6) Any $\mathbb{R}^n$ bundle over paracompact space B can be mapped to γ_n.

Thm Any two bundle maps from E to γ_n is homotopic