Chapter 6

Recap: $Gr(k,n) = \left\{ \begin{matrix} k\text{-dim} \\ \text{subspaces} \end{matrix} \text{ in } \mathbb{R}^n \right\}$

Goal: Cell decomposition $\rightsquigarrow H^*(Gr(k,n); \mathbb{Z}_2)$

① Linear algebra

$V \subset \mathbb{R}^n \rightsquigarrow$ choose a basis $v_1 \dots v_k$

$$k\left\{ \underbrace{\begin{pmatrix} \text{---} & v_1 & \text{---} \\ & \vdots & \\ \text{---} & v_k & \text{---} \end{pmatrix}}_{n} \right. \rightsquigarrow M$$

row operations = change of basis

So: $Gr(k,n) = \left\{ \begin{matrix} \text{rank } k \\ k \times n \text{ matrices} \end{matrix} \right\} \Big/ \begin{matrix} \text{row} \\ \text{operations} \end{matrix}$.

$$M \rightsquigarrow \text{rref} \quad \begin{array}{c} \begin{matrix} \qquad\qquad \sigma_1 & \quad \sigma_2 & \qquad \sigma_k \end{matrix} \\ \begin{pmatrix} * & \cdots & * & 1 & 0 & 0 & 0 & \cdots & & & \\ * & \cdots & & 0 & * & 1 & 0 & \cdots & & & \\ & & & & & 0 & & * & 1 & 0 & \\ 0 & & & 0 & & 0 & & & 0 & * & 1 \end{pmatrix} \end{array}$$

$\text{rank}(M) = k \Rightarrow k$ 1's in positions $\sigma_1 < \sigma_2 < \dots < \sigma_k$

k-element subset of $\{1, \dots, n\}$

rref is unique $\Rightarrow Gr(k,n) = \coprod_{\sigma_1 < \dots < \sigma_k} e(\sigma_1 \dots \sigma_k)$

Schubert cell

$e(\sigma_1 \dots \sigma_k) \simeq \mathbb{R}^d$

$e(\sigma_1,\dots,\sigma_k) \simeq \mathbb{R}$

where $d = (\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_k - k)$

__Ex__ $Gr(2,4)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix} \quad \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}$$

$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$
$d=0$	$d=1$	$d=2$	$d=2$	$d=3$

$$\begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix} \{3,4\} \quad d=4.$$

$\#\text{cells} = \binom{n}{k}$

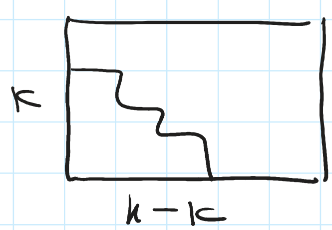
__Exercise__ $\displaystyle\sum_{\underline{\sigma}} q^{d(\underline{\sigma})} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)(1-q)\cdots(1-q^{n-k})}$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{\cancel{(1-q)}\cancel{(1-q^2)}(1-q^3)(1-q^4)}{\cancel{(1-q)}\cancel{(1-q^2)}(1-q)(1-q^2)} \underset{\text{ex.}}{=} 1 + q + 2q^2 + q^3 + q^4$$

__Also:__ $\sigma_1 < \sigma_2 < \dots < \sigma_k$

$\rightsquigarrow (\sigma_1 - 1) \le (\sigma_2 - 2) \le \dots \le (\sigma_k - k)$

partition with $\le k$ parts.



② More abstractly, we consider a __flag__

More abstractly, we consider a flag

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1 \dots e_n \rangle$$

$$\begin{array}{c} v_1 \to \\ v_2 \to \end{array} \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}$$

$\dim V \cap \langle e_1 \rangle = 0$

$\dim V \cap \langle e_1, e_2 \rangle = 1 = \langle v_1 \rangle$

$\dim V \cap \langle e_1, e_2, e_3 \rangle = 1$

$\dim V \cap \langle e_1, e_2, e_3, e_4 \rangle = 2$

More generally, $\dim V \cap \langle e_1 \dots e_m \rangle = \# \{ \sigma_i \leq m \}$

$$\dim V \cap \langle e_1 \rangle \leq \dim V \cap \langle e_1, e_2 \rangle \leq \dots \leq \dim V \cap \langle e_1 \dots e_n \rangle$$

$$\dim V \cap \langle e_1 \dots e_m \rangle = \begin{cases} \dim V \cap \langle e_1 \dots e_{m-1} \rangle \\ \dim V \cap \langle e_1 \dots e_{m-1} \rangle + 1 \leftarrow \text{"jump"} \end{cases}$$

σ_i = <u>sequence of jumps</u>.

(3) Stability: $Gr(k, n) \longrightarrow Gr(k, n+1)$

$$k \Big\{ \Big(\underbrace{\qquad\qquad}_{n} \Big| \underbrace{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}_{1} \Big)$$

cell in $Gr(k,n)$ $e(\sigma)$ $\longrightarrow$ $e(\sigma)$ in $Gr(k, n+1)$

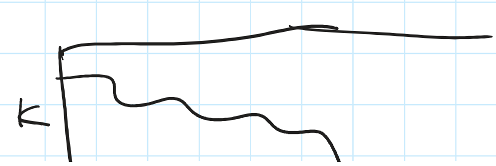
So $Gr(k,n)$ is a CW subcomplex of $Gr(k, n+1)$.

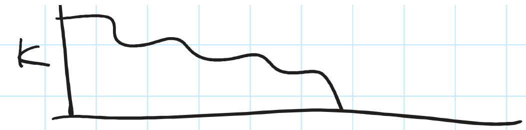
$$Gr(k, \infty) = \lim_{n \to \infty} Gr(k,n)$$

$\Rightarrow$ cell decomposition of $Gr(k, \infty)$

cells $= \{ 1 \leq \sigma_1 < \sigma_2 < \dots < \sigma_k \} =$

$=$ partitions with at most



= partitions with at most k parts. 

<u>Note</u> Same works for $Gr_{\mathbb{C}}(k,n)$ over $\mathbb{C}$.

<u>Thm</u> ① For $Gr(k,n)$, $d=0 \mod 2$

and $H_*(Gr(k,n);\mathbb{Z}_2)$ is generated by cells.

② For $Gr_{\mathbb{C}}(k,n)$, $d=0$ (all cells even dim)

and $H_*(Gr_{\mathbb{C}}(k,n))$ is generated by cells.

<u>Thm</u> ① $H^*(Gr(k,n);\mathbb{Z}_2)$ as a <u>ring</u> is generated by

$w_1=w_1(\gamma_k), \dots w_k=w_k(\gamma_k)$ and

$\bar{w}_1=w_1(\gamma_k^\perp), \dots \bar{w}_{n-k}=w_{n-k}(\gamma_k^\perp)$

mod relations:

$$(1+w_1+\dots+w_k)\cdot(1+\bar{w}_1\dots+\bar{w}_{n-k})=1$$

$\gamma_k \oplus \gamma_k^\perp = \mathbb{R}^n$ trivial

Whitney sum formula

Equivalently,

$$\mathrm{Coef}_{>n-k}\ \frac{1}{1+w_1+\dots+w_k}=0.$$

② $H^*(Gr(k,\infty);\mathbb{Z}_2)=\mathbb{Z}_2[w_1,\dots w_k]$

and no relations.

and no relations.

(3) Same for $Gr_{\mathbb{C}}(k,n)$, $Gr_{\mathbb{C}}(k,\infty)$ and $\mathbb{Z}$ coefs.