Lecture 7 (11/14)

Tuesday, November 14, 2023 8:23 AM

$H^*(Gr)$ as a ring

① Recall: $H^*(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2[a]/a^{n+1}$,

$H^*(\mathbb{RP}^\infty, \mathbb{Z}_2) = \mathbb{Z}_2[a]$, no relations!

γ^1 = tautological line bundle on $\mathbb{RP}^\infty$

$w_1(\gamma^1) = a$ generates H^*.

② By Künneth formula $H^*(\mathbb{RP}^\infty \times \ldots \times \mathbb{RP}^\infty) =$

$= H^*(\mathbb{RP}^\infty) \otimes \ldots \otimes H^*(\mathbb{RP}^\infty)$

$= \mathbb{Z}_2[a_1 \ldots a_k]$ $\gamma^1_{(i)}$ = tautological line bundle on i-th $\mathbb{RP}^\infty$

$\xi := \gamma^1_{(1)} \oplus \ldots \oplus \gamma^1_{(k)}$

By Whitney sum formula,

$w(\xi) = w(\gamma^1_{(1)}) \ldots w(\gamma^1_{(k)}) =$

$= (1+a_1)(1+a_2) \ldots (1+a_k)$

$= 1 + e_1(a) + e_2(a) + \ldots + e_k(a)$

$e_s(a)$ = elementary symmetric functions

$e_1(a) = a_1 + \ldots + a_k$ $e_2(a) = \sum_{i<j} a_i a_j$ etc.

<u>Aside</u> "Splitting principle, part 1"

$V = L_1 \oplus \ldots \oplus L_k$ direct sum of line bdles

$V = \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_k$ direct sum of line bdles

$w_i(V) =$ elementary symmetric fns in $w_1(\mathcal{L}_i)$

$w(V) = (1 + w_1(\mathcal{L}_1))(1 + w_1(\mathcal{L}_2)) \dots (1 + w_1(\mathcal{L}_k))$

<u>Thm</u> $H^*(Gr(k,\infty); \mathbb{Z}_2) = \mathbb{Z}_2[w_1 \dots w_k]$

where $w_i = w_i(\gamma^k)$ tautological k-plane bdle.

<u>Proof</u> (1) There are no relations between w_i

Assume $g(w_1 \dots w_k) = 0$

$\gamma'_{(1)} \oplus \dots \oplus \gamma'_{(k)}$ = rank k bundle on $\mathbb{RP}^\infty \times \dots \mathbb{RP}^\infty$

$\Rightarrow$ there is a bundle map

$$\begin{array}{ccc} \gamma'_{(1)} \oplus \dots \oplus \gamma'_{(k)} & \longrightarrow & \gamma^k \\ \downarrow & & \downarrow \\ \mathbb{RP}^\infty \times \dots \times \mathbb{RP}^\infty & \xrightarrow[f]{} & Gr(k,\infty) \end{array}$$

$f^*(w_i(\gamma^k)) = w_i(f^*\gamma^k) = w_i(\gamma'_{(1)} \oplus \dots \oplus \gamma'_{(k)})$

$=$ elementary symm. in $a_1 \dots a_k$.

If $g(w_1 \dots w_k) = 0$ then

$g[e_1(\underline{a}), \dots e_k(\underline{a})] = 0$

But elementary symmetric functions are algebraically independent! Contradiction, so $g = 0$.

(2) $w_1 \dots w_k$ generate $H^*(Gr(k,\infty))$

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$$\mathbb{Z}_2[w_1 \dots w_k] \xrightarrow[\text{inj}]{} H^*$$

$$\dim H^r \geq \left(\mathbb{Z}_2[w_1 \dots w_k]\right)^{\deg r} = \#(m_1 \dots m_k : m_1 + 2m_2 + \dots + k m_k = r)$$

$$= \#\text{ partitions of } r \text{ with } \leq k \text{ parts.}$$

On the other hand

$$\dim H^r = \dim H_r = \frac{\ker d}{\operatorname{Im} d} \leq \dim(\ker d) \leq \#(r\text{-cells})$$

But $\#(r\text{-dim Schubert cells}) = \#$ partitions of r with $\leq k$ parts.

Conclusion $\dim H^r = \#$ partitions, all inequalities are equalities.

$d = 0 \pmod 2$.

$\square$