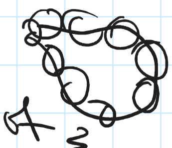
Orientation

Def $\xi \to B$ is orientable, if we can choose orientation in every fiber of ξ such that it is locally constant wrt trivialization of ξ.

Ex M = manifold, M is orientable $\Leftrightarrow TM$ is orientable.

Def We define an orientation cocycle $\theta \in H^1(B;\mathbb{Z}_2)$ as follows: if α is a loop in B then

$$\theta(\alpha) = \begin{cases} 0, & \text{if } \alpha \text{ preserves orientation} \\ 1, & \text{if } \alpha \text{ reverses orientation of } \xi \end{cases}$$



Exercise
- $\theta(\alpha_1 + \alpha_2) = \theta(\alpha_1) + \theta(\alpha_2) \mod 2$
- $\theta(\partial(2\text{-cell})) = 0$

$\Rightarrow \theta$ defines a linear functional $H_1(B;\mathbb{Z}_2) \to \mathbb{Z}_2$.

Thm We have $\theta = w_1(\xi)$.

Sketch of proof: We need to check the following:

(a) On $\mathbb{R}P^n$ and tautol. bundle γ^1,

$$\theta = a \in H^1(\mathbb{R}P^n) \qquad \text{(ex: Möbius band)}$$

Same on $\mathbb{R}P^\infty$. Note: $H_1(\mathbb{R}P^n) = \pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$.

(b) On $(\mathbb{R}P^\infty)^k$ and rank k bundle $\gamma^1_{(1)} \oplus \dots \oplus \gamma^1_{(k)}$

$\theta = a_1 + \dots + a_k \in H^1$, since each of the loops reverses orientation of 1 component.

$\Theta = a_1 + \dots + a_k \in H$, since each of the loops reverses orientation of 1 component.

(c) Prove that Θ is functorial under bdle maps

(d) Use $(\mathbb{R}P^\infty)^k \to Gr(k,\infty)$ to prove thm for $\underset{\gamma^k}{Gr(k,\infty)}$ and then use functoriality & universal property of γ^k.

<u>Cor</u> ξ is orientable iff $w_1(\xi) = 0$.

Thom construction

<u>Observe</u>: Orientation of $\mathbb{R}^n \longleftrightarrow$ class in $H^n(B^n, \partial B^n) \longleftrightarrow H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$

$$H^n(B^n, \partial B^n) \cong H^n(B^n / S^{n-1}) \cong H^n(S^n) \simeq \mathbb{Z}$$

<u>Def</u> Given $\xi : E \to B$ vector bdle with some choice of metric:

$D(E) =$ disk bdle $= \{ v \in E : \langle v, v\rangle \le 1\}$

$S(E) =$ sphere bdle $= \{ v \in E : \langle v, v\rangle = 1\}$

<u>Thom space</u> $T(E) = D(E)/S(E)$

<u>Fact</u> $\tilde{H}^*(E, \underset{\substack{\| \\ E_0 \text{ in book}}}{E \setminus B}) \simeq \tilde{H}^*(D(E), S(E)) \simeq \tilde{H}^*(T(E))$ with any coefs.

<u>Thm</u> (Thom isomorphism). $\xi : E \to B$ rank k vector bdle

(a) $\tilde{H}^i(T(E); \mathbb{Z}_2) \simeq \{ 0$ if $i < k$

(a) $\tilde{H}^i(T(E); \mathbb{Z}_2) \simeq \begin{cases} 0, & \text{if } i < k \\ \tilde{H}^{i-k}(B; \mathbb{Z}_2) & \text{if } i \geq k \end{cases}$

(b) If ξ is orientable then same is true with $\mathbb{Z}$ coefficients.

(c) More precisely, there is a class $u \in H^n(T(E))$ such that $u|_{\text{Fiber}} \in H^n(B^n/S^{n-1}) \simeq \mathbb{Z}$ restricts to the orientation class and

$$H^n(B) \simeq H^n(E) \xrightarrow{\cup u} H^{n+k}(E, E \setminus B)$$

is the Thom isomorphism.

<u>Sketch of proof:</u> (a) σ = n-cell in B

$$\pi : D(E) \xrightarrow{B^k} B \qquad \pi^{-1}(\sigma) = k+n\text{-cell}$$

$$\pi^{-1}(\sigma) \subset D(E), \quad \sigma \subset B, \quad \pi^{-1}(\sigma) \to \sigma$$

$$\partial(\pi^{-1}(\sigma)) = \partial\sigma \times B^k \cup \sigma \times \underbrace{\partial B^k}_{\text{collapsed in } T(E)} \text{ in } D(E)$$

So in $T(E)$ $\pi^{-1}(\sigma) \sim B^{n+k}$

Check: compatible w. boundary maps etc.

If ξ is oriented, orientation on $\sigma \rightsquigarrow$ orientation on $\pi^{-1}(\sigma)$

Assume ξ is oriented, rank k

$$u \in H^k(D(E), S(E)) \longrightarrow H^k(D(E)) \simeq H^k(B)$$

<u>Def</u> The result = <u>Euler class</u> of E.

Thm (a) $\underset{H^k(B,\mathbb{Z})}{\underset{\cap}{eu(E)}} = \underset{H^k(B,\mathbb{Z}_2)}{\underset{\cap}{w_k(E)}} \mod 2$

(b) $eu(E_1 \oplus E_2) = eu(E_1) \cup eu(E_2)$ (w. appropriate orientations)

(c) If M = smooth (resp. orientable) closed n-manifold

then

$$eu(TM) \in H^n(M) \simeq \mathbb{Z} \ (\text{resp. } \mathbb{Z}_2)$$

equals $\chi(M)$ (resp. $\chi(M) \mod 2$)

More precisely, $eu(TM)([M]) = \chi(M)$.

(d) Assume M^n is smooth, ξ = any bundle rank k

s = generic smooth section $Z = \{s = 0\} \subset M$

generic $\to$ s is transverse to M, so Z is smooth. $\dim = n-k$

$$\begin{array}{ccc} [Z] \in H_{n-k}(Z) & \xrightarrow{i_*} & H_{n-k}(M) \\ & & \wr\big\downarrow PD \\ & & H^k(M) \end{array}$$

Then $PD[Z] \simeq eu(\xi)$.

Rmks • (d) $\Rightarrow$ (c): a section of TM is a vector field on M

Z = zero locus of a vector field

$$\sum (-1)^{sgn} = \chi(M)$$

over zeroes of a vector field (w. signs).

$\|$ $[\Delta] \cap [\Delta]$ where Δ is the diagonal in $M \times M$

$[\Delta] \cap [\Delta]$ where Δ is the diagonal in $M \times M$

- $E_1 \oplus E_2$ $\qquad s = (s_1, s_2)$

  s_1, s_2 = generic sections $\quad \{s=0\} = \underbrace{\{s_1 = 0\}}_{Z_1} \cap \underbrace{\{s_2 = 0\}}_{Z_2}$

$PD[Z_1 \cap Z_2] \simeq PD[Z_1] \cup PD[Z_2]$.

- $N \subset M$, ν_N = normal bundle

  section of ν_N = normal vector field ~ perturbation $\widetilde{N}$

  $\{s=0\} = N \cap \widetilde{N}$ ...