Euler Class:

Recall: For $\xi : E \to B$ oriented rank k, there is a class

$$u \in H^k(T(E)) \cong H^k(E, E \setminus B)$$

Such that $u|_{\text{fiber}} \in H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \mathbb{Z}$

restricts to the orientation class, and

$$H^n(B) \simeq H^n(E) \xrightarrow[\simeq]{-\cup u} H^{n+k}(E, E \setminus B)$$

is the Thom isomorphism.

Take the image of u under

$$\begin{tikzcd}
u \in H^k(E, E \setminus B) = H^k(D(E), S(E)) \arrow[d] \arrow[d, bend left] \\
H^k(D(E)) \simeq H^k(B)
\end{tikzcd}$$

and the result is the Euler class of E,

$eu(\xi) \in H^k(B)$.

Thm:

a) $$\underset{\substack{\rotatebox{90}{$\in$}\\ H^k(B,\mathbb{Z})}}{\overset{\substack{eu(E)\\ \shortparallel}}{eu(\xi)}} = \underset{\substack{\uparrow\\ H^k(B,\mathbb{Z}_2)}}{w_k(\xi)} \bmod 2$$

b) $eu(E_1 \oplus E_2) = eu(E_1) \cup eu(E_2)$ (w/ appropriate orientations)

c) If M = smooth closed orientable n-manifold, then

$$eu(TM)([M]) = \chi(M),$$ (essentially $eu(TM)$ equals $\chi(M)$)

non-orientable:
$w_n(TM) = \chi(M) \bmod 2$

d) If M is smooth, ξ is a rank k bundle, s is a generic smooth section $Z = \{s=0\} \subseteq M$, then Z is smooth of $\dim = n-k$, and

$$PD[Z] \simeq eu(\xi)$$

Rmk: (d) $\Rightarrow$ (c), because, if s is a generic section of TM (i.e. a vector field), then Z is the zero locus, and

$$\underset{\substack{\shortparallel\\ [\Delta]\cup[\Delta]}}{\sum_{\substack{\text{zero}\\ \text{locus}}} (-1)^{sgn}} = \chi(M)$$

, where Δ is the diagonal in $M \times M$.

Basically this is the self-intersection number of M with itself, which is one defn. of $\chi(M)$.

• $(d) \to (b)$?: If $E \cong E_1 \oplus E_2$, $s = (s_1, s_2)$ is a general section then

$$\begin{array}{ccccc}\{s=0\} & = & \{s_1=0\} & \cap & \{s_2=0\} \\ \| & & \| & & \| \\ Z & & Z_1 & & Z_2,\end{array}$$

so

$$PD[Z] = PD[Z_1 \cap Z_2] = PD[Z_1] \cup PD[Z_2],$$

Prop: If an oriented v.b. ξ has a nowhere 0 section, then $eu(E)=0$.

Pf: Let $s: B \to E_0$ be an nonzero section, $E_0 = E \setminus B$. Then the composition

$$H^\wedge(B) \xrightarrow{\pi^*} H^\wedge(E) \xrightarrow{i^*} H^\wedge(E_0) \xrightarrow{s^*} H^\wedge(B)$$

is the identity on $H^\wedge(B)$, b.c.

$$B \xrightarrow{s} E_0 \xrightarrow{i} E \xrightarrow{\pi} B$$ is the identity.

By defn, $eu(\xi)$ is the image of $u \in H^\wedge(E, E_0)$,

$$\begin{tikzcd}
u \arrow[r, mapsto] & eu(E) & & & \\
H^\wedge(E,E_0) \arrow[r] \arrow[rrr, bend right] & H^\wedge(B) \arrow[r] & H(E) \arrow[r] & H^\wedge(E_0) \arrow[r] & H^\wedge(B),
\end{tikzcd}$$

but this map is 0 by LES of relative homology.
So the image in $H^\wedge(B)$ is 0, $\Rightarrow eu(E)=0$.
Basically because s 'factors through' E_0 and $eu(E)$ comes from $u \in H^\wedge(E, E_0)$, it gets killed.

(The converse does hold if smooth and rank n by (d)!)

If M is smooth then this is easy by (d).
Unfortunately the converse doesn't hold.

This is one way of thinking about what $eu(E)$ is, it's the obstruction to finding a nowhere 0 section of a bundle.

The book doesn't assume (d), some of these are implied by (d) if M smooth

Cor 11.2: If M is closed in A, $E \to M$ is the normal bundle of M in A, then

$$H^*(E, E_0) \xrightarrow{\cong} H^*(A, A-M)$$

pf: Uses the tubular neighborhood Thm:

- There exists an open neighborhood of M in A which is diffeomorphic to E and is the identity on M, along with excision.

Rmk: this doesn't even depend on the choice of Riemannian metric for A!

! Cor 11.2 implies that the fundamental class $u \in H^k(E, E_0)$ corresponds to a canonical class

$u' \in H^k(A, A-M)$.

(As usual, this is over $\mathbb{Z}_2$, or $\mathbb{Z}$ if orientable)

Thm 11.3: If M is embedded and closed in A, then the natural maps

$$H^k(A, A-M) \to H^k(A) \to H^k(M)$$

take u' to $w_k(\nu_N)$, or if oriented and over $\mathbb{Z}$, take u' to $eu(\nu_N)$.

Pf: This is true by definition for

$$\begin{tikzcd} u \in H^k(E, E_0) \arrow[r] & H^k(E) \arrow[r] & H^k(M) \\ u \arrow[u, phantom, sloped, "\in"] & & \end{tikzcd}$$

then use the natural isomorphism in Cor 11.2.

Defn: The image of u' in $H^k(A)$ is called the dual cohomology class to $M \subseteq A$ of dim. k. If $u' = 0$ then w_k/eu is 0.

Cor 11.4 If $M = M^n$ is smoothly embedded as a closed subset of $\mathbb{R}^{n+k}$, then $w_k(\nu^k) = 0$
($eu(\nu^k) = 0$)

Pf: $u'|_{\mathbb{R}^{n+k}}$ is in $H^k(\mathbb{R}^{n+k}) = 0$.

Alternative perspective

- Let $N^n \subseteq M^{n+k}$, ν_N be the normal bundle.

Sections of ν_N are normal vector fields, i.e. perturbations $\tilde{N}$ of N in M.

$$Z = \{s=0\} = N \cap \tilde{N},$$ i.e the self intersection of N with itself in M.

In R^{n+k}, if M is smoothly embedded and closed we can always perturb it off of itself.

Connection to (d): You can show that the normal bundle ν^n of the diagonal $\Delta \subseteq M \times M$ is canonically isomorphic to TM. So the above agrees with (d).