Characteristic classes, Fall 2023
Homework 1

1. Suppose that $f(x_1, \ldots, x_n)$ is a smooth function on $\mathbb{R}^n$, $X = \{f = 0\}$ and $df \neq 0$ on X.
a) Prove that X is a smooth manifold and find its dimension.
b) Prove that the normal bundle to X in $\mathbb{R}^n$ is trivial.

2. Suppose E is a smooth rank r vector bundle on a smooth n-dimensional manifold M, and $s : M \to E$ is a smooth section. Assume that the graph of s is transverse to the zero section.
a) Define $X = \{x \in M : s(x) = 0\}$. Prove that X is a smooth manifold and find its dimension.
b) Prove that the normal bundle to X in M is isomorphic to the restriction of E to X.
3. Let $S^{2n-1} = \{x_1^2 + \ldots + x_{2n}^2 = 1\}$ be the unit sphere in $\mathbb{R}^{2n}$. Prove that $v(x_1, \ldots, x_{2n}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2n}, -x_{2n-1})$ defines a nowhere zero tangent vector field to S^{2n-1}.
4. a) Let V be a vector space and let $(\cdot,\cdot)_1, \ldots, (\cdot,\cdot)_n$ be a collection of positive definite bilinear forms on V. Suppose that $\lambda_1, \ldots, \lambda_n \geq 0$ and at least one λ_i is nonzero. Prove that

$$(x, y) = \lambda_1(x, y)_1 + \ldots + \lambda_n(x, y)_n$$

is a positive definite bilinear form on V.
b) Prove that any smooth vector bundle on a smooth manifold admits a Euclidean metric. *Hint: use partition of unity and part (a).*