Chern-Weil

Goal: Develope a more computationally friendly way to calculate $c_i(E) \in H^{2i}(M;\mathbb{Z})$

Idea: Use diff. geometry + de Rahm thry:

$$* \; H^{2i}(M;\mathbb{Z}) \hookrightarrow H^{2i}(M;\mathbb{C}) \cong H^{2i}_{deR}(M;\mathbb{C})$$

$\longrightarrow$ want to find a $2i$-form on M, γ_i, which represents the same cohomology class as $c_i(E)$ in $H^{2i}_{deR}(M;\mathbb{C})$

First: We'll develope the general theory of constructing invariants of complex vector bdls from their curvature. (Chern-Weil)

Roadblocks: If we want $c_i(E) = [\gamma_i] \in H^{2i}_{deR}(M;\mathbb{C})$, then

1. γ_i needs to be a GLOBALLY defined diff. $2i$-form on M.

2. γ_i needs to be closed (to represent a non-trivial cohomology class)

3. The construction of γ_i must be independent of connection on E.

Recall: A connection on E is a $\mathbb{C}$-linear map:

$$\nabla : \Gamma(E) \longrightarrow \Gamma(\underbrace{T^*M \otimes E}))$$

elements look locally like: $\sum_i f_i(x)\, dx_i \otimes s_i$

s.t. for $f \in C^\infty(M)$, $\sigma \in \Gamma(E)$

$$\nabla(f\sigma) = \sigma\, df + f \cdot D\sigma$$

Locally: On $U \subset M$, $\exists$ a local frame field $s = (s_1, \dots, s_r)$ where

① $\forall i \in \{1, \dots, r\}$ $s_i \in \Gamma(E|_U)$

② $\forall x \in U$, $(s_i(x), \dots, s_r(x))$ is a basis for $E_x := \pi^{-1}(x)$

$\rightsquigarrow$ Express ∇ locally as a matrix of oneforms as:

$$\nabla s_i = \sum_j s_j \omega_i^j \;;\; \omega_i^j \in A^1|_U$$

$$\omega = [\omega_i^j] = \begin{pmatrix} \omega_1^1 & \omega_2^1 & \cdots & \omega_r^1 \\ \omega_1^2 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \omega_1^r & \cdots & \cdots & \omega_r^r \end{pmatrix} \rightsquigarrow Ds = s \cdot \omega = \left(\sum_j s_j \omega_1^j, \dots, \sum_j s_j \omega_r^j\right)$$

<u>Change of frame field</u>: given another frame field $s' = (s_1', \dots, s_r')$

it's related to s via: $\quad s = s' \cdot a \quad$ * $a : U \longrightarrow GL(r; \mathbb{C})$

Let ω' be ∇ expressed in s': $\quad \omega = a^{-1}\omega' a + a^{-1} da$

$$\left(\boxed{s\omega} = Ds = D(s'a) = D(s')a + s'da = s'\omega' a + s'da = s a^{-1}\omega' a + s a^{-1} da = \boxed{s(a^{-1}\omega' a + a^{-1} da)} \right)$$

*Instead of thinking of ∇ as a map $\Gamma(E)$ to $\Gamma(T^*M \otimes E)$ think of it as $\Gamma(\mathrm{Hom}(E, T^*M \otimes E))$, i.e.

$$\Gamma(T^*M \otimes E \otimes E^*) = \Gamma(T^*M \otimes \mathrm{Hom}(E,E))$$

$$= \Gamma(T^*M \otimes \mathrm{End}(E))$$

"A connection is an End(E) valued 1-form"

For <u>curvature</u>, extend $\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ to a $\mathbb{C}$-linear map

$$\nabla : \Gamma(\Lambda^p T^*M \otimes E) \longrightarrow \Gamma(\Lambda^{p+1} T^*M \otimes E)$$

elements look locally like:

$$\sum_{1 \le i_1 \le \dots \le i_p \le n} f_I(x)\, dx_I \otimes s_i$$

multi-index order p.

s.t. $\forall \sigma \in \Gamma(E),\ \varphi \in \Gamma(\Lambda^p T^*M)$

$$\nabla(\sigma \cdot \varphi) = (\nabla \sigma) \wedge \varphi + (-1)^p \sigma\, d\varphi$$

$C^\infty(M)$ linear

$$\rightsquigarrow \quad \nabla^2 =: R_\nabla : \Gamma(E) \longrightarrow \Gamma(\Lambda^2 T^*M \otimes E)$$

<u>Locally</u>: Given a frame field $(s_1, \dots, s_r)$ on $U \subseteq M$

$$R_\nabla s_i = \nabla^2 s_i = \sum_j \Omega_i^j s_j, \quad \Omega_i^j \in \Lambda^2 T^*M$$

$\therefore$ Locally: $\Omega = [\Omega_i^j] = \begin{pmatrix} \Omega_1^1 & \Omega_2^1 & \cdots & \Omega_r^1 \\ \Omega_1^2 & & & \vdots \\ \vdots & & & \\ \Omega_1^r & \cdots & \cdots & \Omega_r^r \end{pmatrix}$

Local expression via connection form: $\boxed{\Omega = \omega\wedge\omega + d\omega}$ (1)

$*\ s\Omega = D^2 s = D(s\omega) = Ds\wedge\omega + s\,d\omega = s\omega\wedge\omega + s\,d\omega = s(\omega\wedge\omega + d\omega)$

Change of Local Frame: Let $s = s'\cdot a$, then $\boxed{\Omega = a^{-1}\Omega' a}$ (2)

$*\ s\Omega = D^2 s = D^2(s'a) = D(\underbrace{Ds'}_{\Gamma(\Lambda T^*M\otimes E)}\cdot \overset{C^\infty(M)}{\tilde{a}} + s'da) = D^2s'\cdot a - \cancel{Ds'\wedge da} + \cancel{Ds'\wedge da} + \cancel{s'd^2a}$

$= D^2s'a = s'\Omega' a = sa^{-1}\Omega' a$

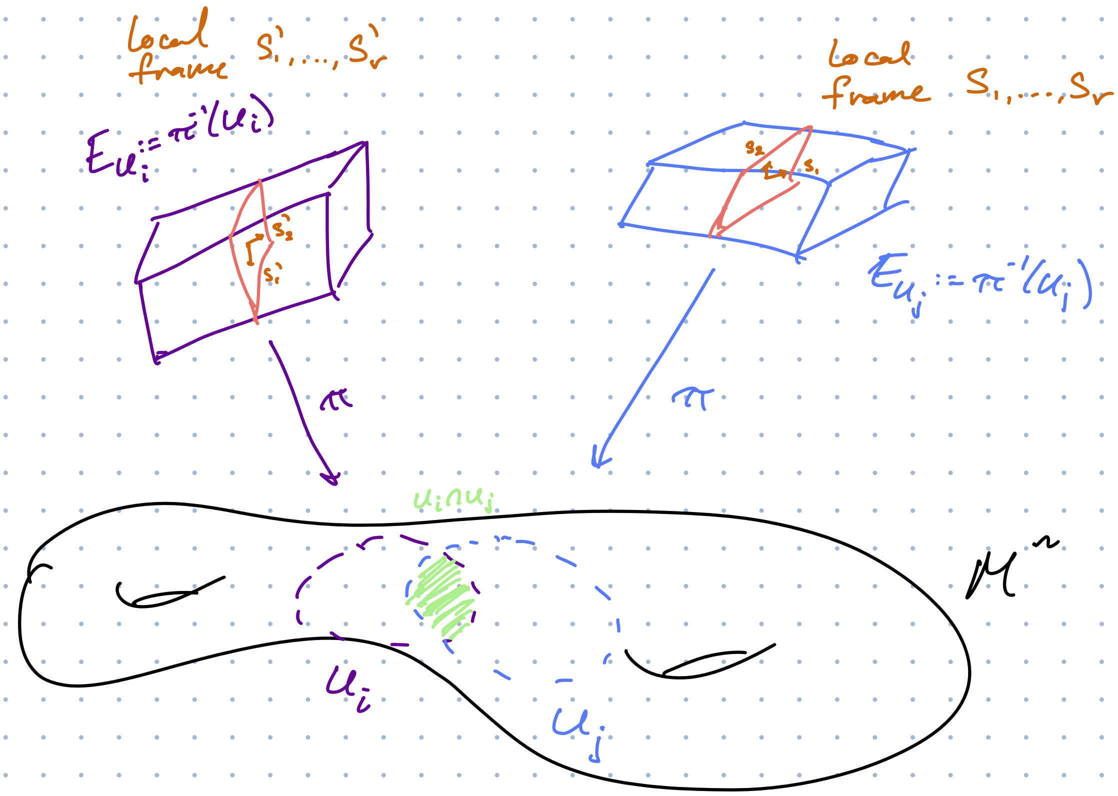
*Instead of thinking of R_∇ as a map $\Gamma(E)$ to $\Gamma(\Lambda^2T^*M\otimes E)$ think of it as $\Gamma(Hom(E, \Lambda^2T^*M\otimes E))$, i.e.

$\Gamma(\Lambda^2T^*M\otimes E\otimes E^*) = \Gamma(\Lambda^2T^*M\otimes Hom(E,E))$

$= \Gamma(\Lambda^2T^*M\otimes End(E))$

"A curvature is an End(E) valued 2-form"

OK now, ... a $2i$-form from curvature... what could go wrong?



Then on $U_i \cap U_j$; transition func: $\varphi_{ij} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \in GL(n;\mathbb{C})$

$\rightsquigarrow$ Curvature transforms as $\Omega = a^{-1}\Omega' a$

$\therefore$ If f was a machine that eats local curvature and spits out a globally well-defined $2i$-form, it must satisfy:

$$\boxed{f(\Omega) = f(a^{-1}\Omega' a)} \;! \qquad a : U_i \cap U_j \longrightarrow GL(r;\mathbb{C})$$

Q: How does structure of transition func. induce codomain of a?

* This now motivates the definition of G-invariant symmetric k-tensors:

Def: $V :=$ vector space (over $\mathbb{C}$). $f : \underbrace{V \times \cdots \times V}_{k} \longrightarrow \mathbb{C}$

f is a symmetric k-tensor if:

(1) $\mathbb{C}$-linear in each input

(2) $\forall X_1, \ldots, X_k \in V$, $f(\ldots, X_i, \ldots, X_j, \ldots) = f(\ldots, X_j, \ldots, X_i, \ldots)$

* f is a linear elmt in $\mathbb{C}[x_1, \ldots, x_k]^{S_k}$

Def: Let $G :=$ group s.t. $G \curvearrowright V$ linearly.

A symmetric k-tensor is G-invariant if $\forall g \in G$, $X \in V$

$$f(g.X) = f(X)$$

*Notice that if G is the structure group of a principle G-bundle; then $G \curvearrowright \mathfrak{g}$ linearly, $\mathfrak{g} := T_I G$ (the lie algebra of G)

$\rightsquigarrow$ for our case think of $G = GL(r; \mathbb{C})$ to describe the "structure"/"transitions" between overlapping local trivializations of E.

i.e. local frame fields are expressed pointwise via

$$\left(s_1(x), \ldots, s_r(x) \right) \in GL(r; \mathbb{C})$$

then change of basis action is via adjoint action $G \curvearrowright \mathfrak{g} = \mathfrak{gl}(r; \mathbb{C})$

$$\Longrightarrow \quad \underset{\mathfrak{gl}(r;\mathbb{C})}{\Omega} = a^{-1} \Omega a$$

ptwise a matrix in $GL(r; \mathbb{C})$

Now, we DESTROY roadblock ① via:

$\boxed{\text{Claim: If } f \in I^k(G) := \{G\text{-invariant symmetric } k\text{-tensors on } \mathfrak{g}\}\text{, then } f(R_\nabla) \text{ is a GLOBALLY defined } 2i\text{-form on } M}$

Notation: $f \in I^k(G)$; $f(X) := f(\underbrace{X, \dots, X}_{k})$

* by $f(R_\nabla)$ I specify it's local expression on $U \subset M$ via $f(\Omega)$ where Ω is the matrix of 2-forms on M from expression of R_∇ in terms of a local frame field $s_1, \dots, s_r$ on E_U

Notation: Notice $f: \mathfrak{g} \times \dots \times \mathfrak{g} \longrightarrow \mathbb{C}$, but Ω is a $\mathfrak{gl}(r;\mathbb{C})$ matrix w/ 2-form entries...

Locally, $R_\nabla \in \Lambda^2 T^*M \otimes \mathrm{End}(E)$; i.e. $\Omega = \mu \otimes \ell$, $\mu \in \Lambda^2 T^* M$, $\ell \in \mathrm{End}(E)$

$\therefore$ define $f(\Omega) := f(\mu \otimes \ell, \dots, \mu \otimes \ell)$

$$= \underbrace{(\mu \wedge \dots \wedge \mu)}_{\substack{\text{a } 2k\text{-form} \\ \text{on } U \subset M}} \underbrace{f(\ell)}_{\substack{\text{smth. func.} \\ f: V \times \dots \times V \to \mathbb{C}}} \in \Omega^{2k}(U; \mathbb{C})$$

f is $G = GL(r;\mathbb{C})$ invariant $\Rightarrow$ on intersecting charts $U \cap U'$

$$\underline{\underline{f(\Omega) = f(a^{-1}\Omega' a)}}$$

$\therefore f(R_\nabla)$ is globally defined $2k$-form on M

What about roadblocks ② & ③?

*Notice so far we only used G-invariance and linearity... not symmetry yet...

<u>Thm (Chern-Weil)</u> If $f \in I^k(G)$

i) $f(R_\nabla)$ is a closed $2k$-form on M

* $[f(R_\nabla)] \in H^{2k}(M;\mathbb{C})$

ii) $[f(R_\nabla)]$ is independent of choice of ∇

* It is an invariant of E

iii) $\exists$ a ring homomorphism $CW: I^*(G) \longrightarrow H^*(M;\mathbb{C})$
$$f \longmapsto [f(R_\nabla)]$$

* $[f(R_\nabla)]$ is a characteristic cohomology class of E corresponding to f.

<u>Let's prove i), ii)</u>

pf: i) Let $f \in I^k(G)$, $R_\nabla :=$ Curvature of E

$\Omega :=$ Local curvature of E on $U \subset M$

Lemma: Let $X, X_1, \ldots, X_k \in \mathfrak{g}$, then

$$f([X,X_1], X_2, \ldots, X_k) + f(X_1, [X,X_2], \ldots, X_k) + \ldots + f(X_1, X_2, \ldots, [X,X_k]) = 0$$

pf of Lemma: Since f symmetric

since $f(\tilde{X}) = f(e^{tX} \underbrace{\tilde{X}}_{\in \mathfrak{g} = T_I G} e^{-tX})$

$$0 = \frac{d}{dt}\Big|_{t=0} f(e^{tX} X_1 e^{-tX}, e^{tX} X_2 e^{-tX}, \ldots, e^{tX} X_k e^{-tX}) \Big\}$$ this is a constant func of t

$$= f\left(\frac{d}{dt}\Big|_{t=0} e^{tX} X_1 e^{-tX}, X_2, \ldots, X_k\right) + f\left(X_1, \frac{d}{dt}\Big|_{t=0} e^{tX} X_2 e^{-tX}, \ldots, X_k\right)$$

$$+ \ldots + f\left(X_1, X_2, \ldots, \frac{d}{dt}\Big|_{t=0} e^{tX} X_k e^{-tX}\right)$$

$$* \ \frac{d}{dt}\left[e^{tX} X_i e^{-tX}\right]\Big|_{t=0} = \left[X e^{tX} X_i e^{-tX} - e^{tX} X_i X e^{-tX}\right]\Big|_{t=0} = XX_i - X_i X = [X, X_i]$$

$$= f([X,X_1], X_2, \ldots, X_k) + f(X_1, [X,X_2], \ldots, X_k) + \ldots + f(X_1, X_2, \ldots, [X,X_k]) \qquad \square$$

pf of i): Locally $df(R_\nabla) = df(\Omega)$

$$df(\Omega) = df(\Omega, \ldots, \Omega)$$

$$= f(d\Omega, \Omega, \ldots, \Omega) + f(\Omega, d\Omega, \ldots, \Omega) + \ldots + f(\Omega, \Omega, \ldots, d\Omega)$$

$$= f([\Omega,\omega], \Omega, \ldots, \Omega) + f(\Omega, [\Omega,\omega], \ldots, \Omega) + \ldots + f(\Omega, \Omega, \ldots, [\Omega,\omega])$$

$$= 0$$ (by Lemma)

Bianchi Identity:

$$dR_\nabla = [R_\nabla, \omega] = R_\nabla \wedge \omega - \omega \wedge R_\nabla$$

pf: $\boxed{R_\nabla = d\omega + \omega \wedge \omega}$

$$dR_\nabla = d(d\omega + \omega\wedge\omega) = \cancel{d^2}\omega + d(\omega \wedge \omega)$$

$$= d\omega \wedge \omega - \omega \wedge d\omega$$

$$= d\omega \wedge \omega + \omega\wedge\omega\wedge\omega - \omega \wedge d\omega - \omega\wedge\omega\wedge\omega$$

$$= (d\omega + \omega\wedge\omega)\wedge\omega - \omega\wedge(d\omega + \omega\wedge\omega)$$

$$= R_\nabla \wedge \omega - \omega \wedge R_\nabla = [R_\nabla, \omega]$$

$$\therefore \boxed{df(R_\nabla) = 0 \Rightarrow f(R_\nabla) \text{ exact}}$$

$\square$

ii) We gotta get HOMOLOGICAL *Let's Go!!!

Lemma 1: If $f \simeq g$ are homotopic maps $X \longrightarrow Y$ (top spaces), the induced chain maps $f^*, g^* : \Omega^k(Y) \longrightarrow \Omega^k(X)$ are chain homotopic i.e. $\{Q : \Omega^k(Y) \longrightarrow \Omega^{k-1}(X)\}$ s.t.

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \Omega^{k-1}(Y) \arrow[r, "d"] & \Omega^k(Y) \arrow[r, "d"] \arrow[dl, "Q"'] \arrow[d, shift right, "f^*"'] \arrow[d, shift left, "g^*"] & \Omega^{k+1}(Y) \arrow[r, "d"] \arrow[dl, "Q"] & \cdots \\
\cdots \arrow[r, "d"] & \Omega^{k-1}(X) \arrow[r, "d"] & \Omega^k(X) \arrow[r, "d"] & \Omega^{k+1}(X) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

$\forall \omega \in \Omega^k(Y)$, $\forall k$

$$f^*\omega - g^*\omega = d(Q(\omega)) + Q(d\omega)$$

Lemma 2: If two chain maps are homotopic, then their induced maps on cohomology are equal, i.e. if $f^*, g^* : \Omega^k(Y) \longrightarrow \Omega^k(X)$ are chain homotopic, $\forall \omega \in \Omega^k(Y)$:

$$[f^*\omega] = [g^*\omega] \in H^k_{deR}(X; \mathbb{C})$$

pfs: Standard pf for singular homology in 215B

- - - - cohomology in 215C?

$\rightsquigarrow$ Hatcher ch 2/3 or any Homological Algebra text

pf of ii) Let $i_0 : M \longrightarrow M \times \mathbb{R}$, $x \longmapsto (x, 0)$ $\quad$ $i_1 : M \longrightarrow M \times \mathbb{R}$, $x \longmapsto (x, 1)$

Clearly $i_0 \simeq i_1$ are homotopic via $F_t : M \longrightarrow M \times \mathbb{R}$, $x \longmapsto (x, t)$

Lemmas ① + ② $\Rightarrow$ their induced maps on cohomology are equal, i.e. $[i_0^*\omega] = [i_1^*\omega] \in H^k_{deR}(M; \mathbb{C})$ $\forall \omega \in \Omega^k(M \times \mathbb{R})$

Now let ∇_0 and ∇_1 be two connections on E.
Locally on $U \subset M$ in a local frame field we can express ∇_0, ∇_1 via matrices of 1-forms w/ values in $\mathrm{End}(E)$: $\omega_0, \omega_1 \in \Omega^1(U) \otimes \mathrm{End}(E)$

Consider $\tilde{\omega} = (1-s)\omega_0 + s\omega_1 \in \Omega^1(U \times \mathbb{R}) \otimes \mathrm{End}(E)$; $s \in [0,1]$ (*)

$\tilde{\omega}$ defines a connection on the vector bundle $E \times \mathbb{R}$ over $M \times \mathbb{R}$ in the natural way: $\tilde{\nabla}$ can be constructed globally on $E \times \mathbb{R}$ from the local data $\tilde{\omega}$ via the canonical projection $p: M \times \mathbb{R} \longrightarrow M$.

*The pullback p^*E is a rank r vector bdl over $M \times \mathbb{R}$ where $p^*\nabla_0$ and $p^*\nabla_1$ are connections on p^*E. Then $\tilde{\nabla} = (1-s)\nabla_0 + s\nabla_1$ is a linear connection on p^*E

$\therefore$ Since $i_0^* R_{\tilde{\nabla}} = R_{\tilde{\nabla}} \circ i_0 = R_{\nabla_0}$; $i_1^* R_{\tilde{\nabla}} = R_{\tilde{\nabla}} \circ i_1 = R_{\nabla_1}$

$\Longrightarrow$ for $f \in I^k(G)$, $\quad i_0^* f(R_{\tilde{\nabla}}) = f(R_{\nabla_0})$
$\quad i_1^* f(R_{\tilde{\nabla}}) = f(R_{\nabla_1})$

$i_0 \simeq i_1$ homotopy equiv. $\rightsquigarrow [i_0^* f(R_{\tilde{\nabla}})] = [i_1^* f(R_{\tilde{\nabla}})] \in H^{2k}_{dR}(M, \mathbb{C})$

$$\Longrightarrow \boxed{[f(R_{\nabla_0})] = [f(R_{\nabla_1})]}$$

$\therefore [f(R_\nabla)] \in H^{2k}_{deR}(M; \mathbb{C})$ is independent of connection ∇ $\square$

$\rightsquigarrow$ $\therefore$ given $f \in I^k(G)$, denote $f(E) := [f(R_\nabla)] \in H^{2k}_{deR}$

Fact: $f(E)$ is indeed an invariant, i.e. if F is another vector bdl, s.t. $E \cong F$ then $f(E) = f(F)$.

(i.e. $\pi_1 : E \to M$, $\pi_2 : F \to N$, $E \cong F$ if $\exists$ $f : E \to F$, $g : M \to N$ s.t.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circlearrowleft"] & F \arrow[d, "\pi_2"] \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

)

* idea: $E \cong F$ induces a ptwise isom of the fibers $\pi_1^{-1}(x) \underset{\varphi}{\cong} \pi_2^{-1}(x)$ $\left(\begin{matrix} \pi_1 : E \to M \\ \pi_2 : F \to M \end{matrix}\right)$

$\Rightarrow$ Connections on E and F can be canonically identified via this fiber-wise isom. locally

say $s = (s_1, \dots, s_r)$ a local frame field for $E|_U$, then $\varphi s = (\varphi s_1, \dots, \varphi s_r)$ a local frame field for $F|_U$

and the connection form $\nabla_E s = s \cdot \omega \longleftrightarrow \nabla_F \varphi s = \varphi(\nabla_E s)$