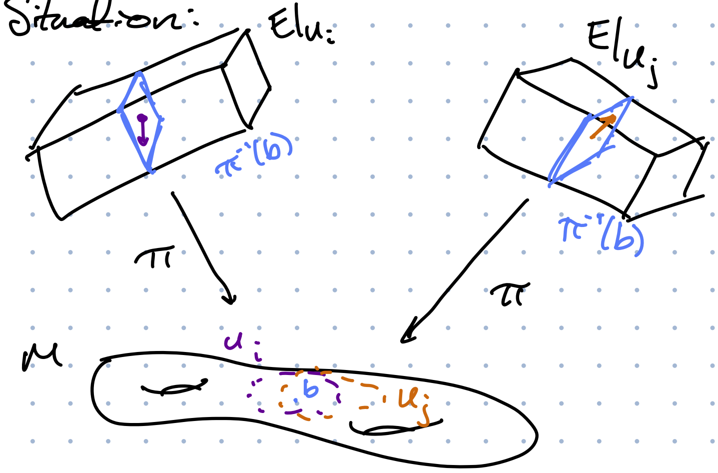
Recall: Last Time; Characteristic deRahm classes of M from Curvature $R_\nabla \in \Gamma(\Lambda^2 T^*M \otimes End(E))$

Situation:



$G = GL(r;\mathbb{C})$ the structure group of E, i.e. the place where transition func. of E live.

* Take $U_i, U_j \subset M$ open s.t. $\pi: E \to M$ is trivial over U_i, U_j.

Charts: $\varphi_{U_i}: U_i \times \mathbb{C}^r \longrightarrow \pi^{-1}(U_i)$

$\varphi_{U_j}: U_j \times \mathbb{C}^r \longrightarrow \pi^{-1}(U_j)$

Transition: $\varphi_{U_i}^{-1} \circ \varphi_{U_j} : (U_i \cap U_j) \times \mathbb{C}^r \longrightarrow (U_i \cap U_j) \times \mathbb{C}^r$

$(x, v) \longmapsto (x, a(x)\cdot v)$

where $a: U_i \cap U_j \longrightarrow \underbrace{GL(r;\mathbb{C})}$

How we identify fibers of pts in $U_i \cap U_j$; an Endomorphism of $\pi^{-1}(b) \cong \mathbb{C}^r$ for each $b \in U_i \cap U_j$.

Then $V := End(E)$, in particular, endomorphisms of the fibers;

$End(\mathbb{C}^r) = \underset{\uparrow}{gl(r;\mathbb{C})} =: V$

$r \times r$ matrices.

Where since $V = T_I G$, $G \curvearrowright V$ linearly via conjugation:

$$\begin{aligned} G \times V &\longrightarrow V \\ (g.A) &\longmapsto gAg^{-1} \end{aligned}$$

$$\begin{tikzcd}
\rightsquigarrow I^k(G) := \left\{\begin{matrix}\text{G-inv. symmetric} \\ \text{k-tensors}\end{matrix}\right\} \arrow[r, leftrightarrow, squiggly] \arrow[d, phantom, "\cup"] & P^k(G) := \left\{\begin{matrix}\text{G-inv.} \\ \text{degree } k \\ \text{homogeneous} \\ \text{polynomials on } V\end{matrix}\right\} \arrow[d, phantom, "\cup"] \\
\left\{\begin{matrix} f : V\times\cdots\times V \longrightarrow \mathbb{C} \\ (x_1,\ldots,x_k) \longmapsto f(x_1,\ldots,x_k)\end{matrix}\right\} \arrow[r, leftrightarrow] & \left\{\begin{matrix} \tilde{f} : V \longrightarrow \mathbb{C} \\ v \longmapsto f(v,\ldots,v)\end{matrix}\right\}
\end{tikzcd}$$

* $\tilde{f}$ homogeneous deg k

since $\forall t \in \mathbb{C}$:

$$\underline{\tilde{f}(tv)} = f(tv,\ldots,tv) = t^k f(v,\ldots,v)$$
$$\underline{= t^k \tilde{f}(v)}$$

Clearly f is G-inv $\Longleftrightarrow$ $\tilde{f}$ G-inv.

Main Result from Last Week

Chern-Weil: Given $f \in I^k(G)$; $R_\nabla \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}\,E)$

Then $\tilde{f}(R_\nabla)$; interpreted locally as $\Omega \otimes A \in \Lambda^2 T^*M \otimes \mathrm{End}\,E)$

$$\underline{\tilde{f}(\Omega\otimes A)} = f(\Omega\otimes A, \ldots, \Omega\otimes A)$$
$$= (\Lambda^k \Omega)\, f(A,\ldots,A)$$
$$= \underline{(\Lambda^k\Omega)\tilde{f}(A)}$$

① $\tilde{f}(R_\nabla) \in \Omega^{2k}(M;\mathbb{C})$ globally defined $\quad$ ($\tilde{f}(\Omega) = \tilde{f}(a\Omega a^{-1})$)

② $d\tilde{f}(R_\nabla) = 0$, i.e. $[\tilde{f}(R_\nabla)] \in H^{2k}_{deR}(M;\mathbb{C})$ $\quad$ (used Bianchi identity, G-inv, symmetry)

③ $[\tilde{f}(R_\nabla)] \in H^{2k}_{deR}(M;\mathbb{C})$ is independent of ∇ $\quad$ (used homotopic maps i_0, i_1 to induce equivalence on cohomology.)

Ok... cool beans. What abt Chern classes?

$$c_k(E) \in H^{2k}_{dR}(M;\mathbb{C}) \longleftrightarrow H^{2k}(M;\mathbb{Z})$$

Idea: Generate $\tilde{\gamma}_k \in P^k(G)$ s.t. $[\tilde{\gamma}_k(R_\nabla)] = c_k(E) \in H^{2k}_{dR}(M;\mathbb{C})$

Lemma: $$\det(tI_r - A) = \underset{(*)}{\prod_{k=1}^{r}(t-\lambda_k)} = \sum_{k=0}^{r}(-1)^k \underbrace{e_k(\lambda_1,\dots,\lambda_r)}_{\text{Denoted as } e_k(A)} t^{r-k}$$

the characteristic polynomial should vanish @ eigenvalues of A.

"k-th degree elementary symm. polynomials on eigenvalues of A"

Use "Vieta's Formulas"

*Vieta's Formula: Writes coefficients of a polynomial in terms of its roots; Given $P(x) = a_r x^r + a_{r-1}x^{r-1} + \dots + a_1 x + a_0$, w/ complex roots given by: $\lambda_1, \dots, \lambda_r$ (Fund. Thm. of Alg.)
The coefficients of $P(x)$ can be recovered via:

$$(-1)^k \frac{a_{n-k}}{a_n} = e_k(\lambda_1,\dots,\lambda_r)$$

* use this on (*)
⇒ Since $a_n = 1$

* Review: Def of elementary symmetric poly

$e_1(\lambda_1,\dots,\lambda_r) = \lambda_1 + \dots + \lambda_r = \sum_{1\le j\le n} \lambda_j$

$e_2(\lambda_1,\dots,\lambda_r) = \sum_{1\le j<k\le n} \lambda_j\lambda_k$

⋮

$e_k(\lambda_1,\dots,\lambda_r) = \sum_{1\le j_1<\dots<j_k\le n} \lambda_{j_1}\cdots\lambda_{j_k}$

ex: $e_0(\lambda_1,\lambda_2,\lambda_3) = 1$

$e_1(\lambda_1,\lambda_2,\lambda_3) = \lambda_1+\lambda_2+\lambda_3$

$e_2(\lambda_1,\lambda_2,\lambda_3) = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3$

$e_3(\lambda_1,\lambda_2,\lambda_3) = \lambda_1\lambda_2\lambda_3$

In the case of smth rank r vect bdls: $\pi : E \longrightarrow M$

$G := GL(r;\mathbb{C})$. $V := T_I G = \mathfrak{gl}(r;\mathbb{C})$

Since $\det(kV) = k^r(\det V)$

$\forall A \in \mathfrak{gl}(r;\mathbb{C})$ it follows that:

$$\underline{\det\left(\lambda I_r - \frac{1}{2\pi i} A\right)} = \frac{1}{(2\pi i)^r} \det(2\pi i \lambda I_r - A)$$

$$= \frac{1}{(2\pi i)^r} \sum_{k=0}^{r} (-1)^k e_k(A) (2\pi i \lambda)^{r-k}$$

$$= \underline{\sum_{k=0}^{r} (-1)^k \frac{e_i(A)}{(2\pi i)^k} \lambda^{r-k}}$$

$\boxed{\text{Claim 1:}}$ $\gamma_k(A) = (-1)^k \frac{e_i(A)}{(2\pi i)^k}$ for $A \in V = \mathfrak{gl}(r;\mathbb{C})$, is a G-inv deg k homogeneous polynomial. ($\therefore$ Chern-Weil Thry holds)

$\boxed{\text{Claim 2:}}$ $[\gamma_k(R_\nabla)] \in H^{2k}_{deR}(M;\mathbb{C})$ are the axiomatic Chern-classes.

$\boxed{\text{Claim 1}}$ Clearly for each $k \in \{1, \dots, r\}$, $e_K \in P^k := \left\{ \begin{matrix} \text{deg } k \\ \text{homogeneous poly} \\ \text{on } V = \mathfrak{gl}(r; \mathbb{C}) \end{matrix} \right\}$

$$\Downarrow$$

$$e_K(A) := e_K(\lambda_1, \dots, \lambda_r) = \underbrace{\sum_{1 \le j_1 < \cdots < j_K \le n}' \lambda_{j_1} \cdots \lambda_{j_K}}_{\text{homogeneous deg } k \text{ on } \mathbb{C}^r}$$

G-invariant? Notice $tr(A) = \lambda_1 + \dots + \lambda_r$

$e_K(A)$ in terms of trace:

$e_0(A) = 1$

$e_1(A) = \sum_{j=1}^{r} \lambda_j = Tr(A)$

$$e_2(A) = \sum_{1 \le j_1 < j_2 \le n} \lambda_{j_1}\lambda_{j_2} = \frac{e_1(A)^2 - e_1(A^2)}{2}$$

$$= \frac{(Tr(A))^2 - Tr(A^2)}{2}$$

$(e_1(A))^2 = (\Sigma^r \lambda_j)(\Sigma^r \lambda_j)$

$= (\Sigma^r \lambda_j^2) + 2(\sum_{1 \le j_1 < j_2 \le r} \lambda_{j_1}\lambda_{j_2})$

$= e_1(A^2) + 2(e_2(A))$

Lem: If $\lambda_1, \dots, \lambda_r$ eigenvalues of A then $\lambda_1^k, \dots, \lambda_r^k$ eigenvalues of A^k

* Want to generalize this process to $(e_1(A))^k$

$$e_3(A) = \sum_{1 \le j_1 < j_2 < j_3 \le n}' \lambda_{j_1}\lambda_{j_2}\lambda_{j_3} = \frac{-(Tr(A))^3 + 3Tr(A^2)Tr(A) - 2Tr(A^3)}{6}$$

*Insight into generalization:

$$(Tr(A))^3 = (e_1(A))^3 = -2\overbrace{Tr(A^3)}^{e_1(A^3)} + 3\overbrace{Tr(A^2)Tr(A)}^{e_1(A^2)e_1(A)} - 6e_3(A)$$

$$= (-1)^3 \cdot \underbrace{|L_3|}_{\#\text{ 3-cycles in } S_3} Tr(A^3) + (-1)^2 \cdot \underbrace{|L_2|}_{\#\text{ 2-cycles in } S_3} Tr(A^2)Tr(A) + (-1)^3 \cdot 3!\, e_3(A)$$

$$\therefore (e_1(A))^k = (-1)^k \cdot |L_k| \cdot Tr(A^k) + \left(\sum_{j=1}^{k-2} (-1)^{k-j} |L_{k-j}| Tr(A^{k-j}) Tr(A^j) \right)$$

$$+ (-1)^k \cdot k!\, e_k(A)$$

$$\Rightarrow \boxed{ e_k(A) = \frac{\left(\sum_{j=0}^{k} (-1)^{-j-1} \overbrace{|L_{k-j}|}^{\# k-j \text{ cycles in } S_k} Tr(A^{k-j}) Tr(A^j) \right)}{k!} },$$

Plugging in curvature $\Omega = [\Omega_j^i]^{\leftarrow r\times r}$ $\qquad \gamma_k(A) = (-1)^k \dfrac{e_i(A)}{(2\pi i)^k}$

$$\gamma_0(\Omega) = 1$$

$$\gamma_1(\Omega) = -\frac{1}{2\pi i} tr(\Omega) = -\frac{1}{2\pi i} \sum_{i=1}^{r} \Omega_i^i$$

$$\gamma_2(\Omega) = \frac{1}{2!}\left(\frac{1}{2\pi i}\right)^2 \left(tr(\Omega)^2 - tr(\Omega^2) \right) = \frac{1}{2!}\left(\frac{1}{2\pi i}\right)^2 \left(\sum_{1\le i<j\le n} \Omega_i^i \wedge \Omega_j^j - \Omega_j^i \wedge \Omega_i^j \right)$$

$$\gamma_3(\Omega) = \frac{1}{3!}\left(\frac{1}{2\pi i}\right)^3 \left(tr(\Omega)^3 \overset{\swarrow \text{2-cycles odd}}{-} \underset{\nwarrow \# \text{2-cycles in } S_3}{3} tr(\Omega^2) tr(\Omega) \overset{\swarrow \text{3-cycles even}}{+} \underset{\nwarrow \# \text{3-cycles in } S_3}{2} tr(\Omega^3) \right)$$

$$= \frac{1}{3!}\left(\frac{1}{2\pi i}\right)^3 \left(\sum_{1\le i_1<i_2<i_3\le n} \sum_{\sigma\in S_3} sgn(\sigma) \Omega^{i_1}_{i_{\sigma(1)}} \wedge \Omega^{i_2}_{i_{\sigma(2)}} \wedge \Omega^{i_3}_{i_{\sigma(3)}} \right)$$

$$\vdots$$

$$\gamma_r(\Omega) = \frac{1}{r!}\left(\frac{i}{2\pi}\right)^r det(\Omega)$$

$$(k<r) \qquad \gamma_k(\Omega) = \frac{1}{k!}\left(\frac{1}{2\pi i}\right)^k \left(\sum_{1\le i_1<\dots<i_k\le n} \sum_{\sigma\in S_k} sgn(\sigma) \Omega^{i_1}_{i_{\sigma(1)}} \wedge \Omega^{i_2}_{i_{\sigma(2)}} \wedge \dots \wedge \Omega^{i_k}_{i_{\sigma(k)}} \right)$$

all G-invariant as homogeneous poly on V. ✓

$\boxed{\text{Claim 2}}$ From Claim 1 $\Rightarrow [\gamma_k(R_\nabla)] \in H^{2k}_{deR}(M; \mathbb{C})$

To show $[\gamma_k(R_\nabla)] = c_k(E)$ it suffices to show the cohomology classes $\{[\gamma_k(R_\nabla)]\}_{k=0}^{r}$ satisfy the Chern class axioms since Chern Classes are unique $(*)$

$\boxed{\text{Axiom 1}}$ $[\gamma_0(R_\nabla)] = 1$ and $[\gamma_k(R_\nabla)] \in H^{2k}(M; \underline{\underline{\mathbb{Z}}})$

Clearly $[\gamma_0(R_\nabla)] = 1 \in H^0(M; \mathbb{Z})$.

wts $[\gamma_k(R_\nabla)]$ is an <u>INTEGRAL</u> cohomology class

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Thm: (DeRahm) $H^{2k}(M; \mathbb{C}) \cong H^{2k}_{deR}(M; \underline{\underline{\mathbb{C}}})$

smooth-singular cohomology

de-Rahm cohomology

Recall: $$H^{2k}_{deR}(M;\mathbb{C}) := \frac{\operatorname{Ker}(d: \Omega^{2k}(M;\mathbb{C}) \longrightarrow \Omega^{2k+1}(M;\mathbb{C}))}{\operatorname{Im}(d: \Omega^{2k-1}(M;\mathbb{C}) \longrightarrow \Omega^{2k}(M;\mathbb{C}))}$$

$$H^{2k}(M;\mathbb{C}) = \frac{\operatorname{Ker}(\partial^*: C^{2k}(M) \longrightarrow C^{2k+1}(M))}{\operatorname{Im}(\partial^*: C^{2k-1}(M) \longrightarrow C^{2k}(M))}$$

$C_{2k}(M) = \{$ Free gp. generated by smth 2k simplices $\rightarrow \sigma: \Delta_{2k} \xrightarrow[\text{smth}]{} M \}$

$C^{2k}(M) = \operatorname{Hom}(C_{2k}(M), \underline{\underline{\mathbb{C}}})$ *just group homs.

$\partial^* \leftarrow$ precompose w/ $\partial: C_{2k+1}(M) \rightarrow C_{2k}(M)$

$$\begin{array}{ccc} \partial^*: C^{2k}(M) & \longrightarrow & C^{2k+1}(M) \\ \cup\!\! \mid & & \cup\!\! \mid \\ \{f: C_{2k}(M) \to \mathbb{C}\} & \longmapsto & \{f \circ \partial: C_{2k+1}(M) \to C_{2k}(M) \to \mathbb{C}\} \\ \sigma \longmapsto f(\sigma) & & \end{array}$$

The explicit iso:

$$\begin{tikzcd}
\Phi : H^{2K}_{deR}(M;\mathbb{C}) \arrow[r] & H^{2K}(M;\mathbb{C}) \arrow[d, phantom, "\cong" sloped] \\
 & \mathrm{Hom}(H_{2K}(M),\mathbb{C}) \\
{[\omega]} \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & \left\{ \begin{array}{ccc} f : H_{2K}(M) & \longrightarrow & \mathbb{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ {[c:\Delta_{2K}\to M]} & \longmapsto & \int_c \omega \end{array} \right\}
\end{tikzcd}$$

Where on simplices: $\sigma : \Delta_{2K} \to M$

$$\int_\sigma \omega = \int_{\Delta_{2K}} \sigma^* \omega$$

and on chains: $c = \sum_{i=1}^{\ell} c_i \sigma_i$

$$\int_c \omega = \sum_{i=1}^{\ell} c_i \int_{\Delta_{2K}} \sigma_i^* \omega$$

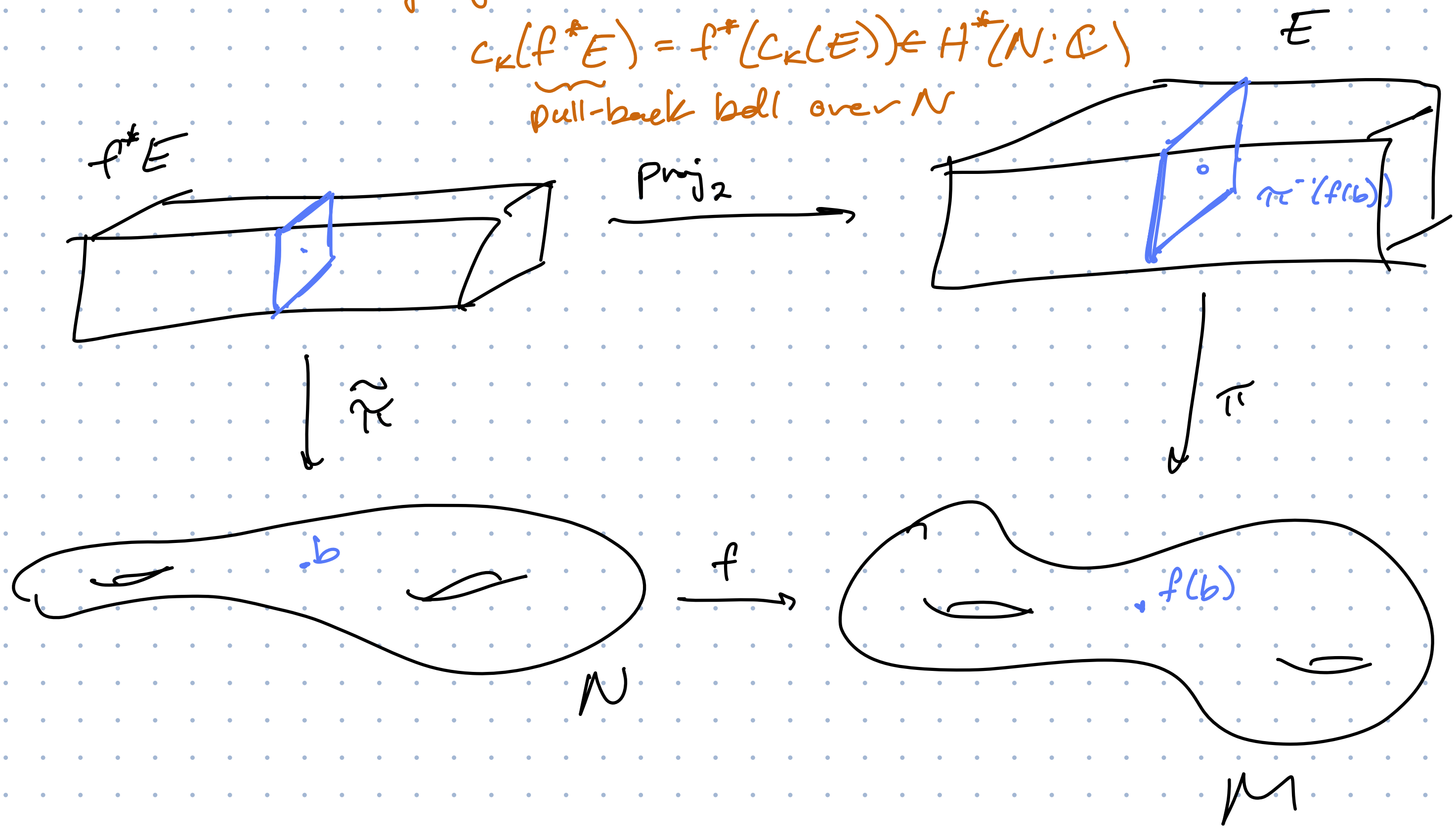
$\therefore$ To show $\gamma_K(R_\nabla)$ is an integral cohomology class, need to make sure: $\forall$ simplices $c : \Delta_{2K} \to M$

$$\begin{array}{c} \int_c \gamma_K(R_\nabla) \in \mathbb{Z} \\ \parallel \\ \sum_{i=1}^{\ell} c_i \int_{\Delta_{2K}} \sigma_i^*(\gamma_K(R_\nabla)) \in \mathbb{Z} \end{array}$$

This is hard...

$\boxed{\text{Axiom 2:}}$ Naturality: given $f: N \longrightarrow \overset{\overset{E}{\downarrow \pi}}{M}$, then

$$\underbrace{c_k(f^*E)}_{\text{pull-back bdl over } N} = f^*(c_k(E)) \in H^*(N; \mathbb{C})$$



$$f^*E = \{(b,v) \in N \times E : b \in N,\ v \in \pi^{-1}(f(b))\}$$

If ∇ a connection on E, we see $f^*\nabla = \nabla \circ f$ is a connection on f^*E w/ curvature f^*R_∇.

If ∇ written locally on $\overset{\in M}{U}$ as $[\omega_i^j]$ in terms of local frame $s = (s_1, \dots, s_r)$; $\nabla s_j = \sum_{i=1}^{r} s_i \omega_i^j$

$\therefore$ $f^*s = (f^*s_1, \dots, f^*s_r)$ a local frame on f^*E. over $f^{-1}(U) \subseteq N$ and $\therefore$ $(f^*\nabla)(f^*s_j) := f^*\left(\sum_{i=1}^{r} \omega_i^j s_i\right) = \sum_{i=1}^{r} \underbrace{f^*(\omega_i^j)}_{\Gamma(T^*N)}\,.\, \underbrace{f^*(s_i)}_{\Gamma(f^*E)}$

$\therefore$ Locall connection on f^*E induced by ∇ is:

$$[f^*(\omega_i^j)]_{ij}$$

Same reason $\Rightarrow$ local curvature of f^*E is $f^*\Omega$.

$$\therefore\ c_k(f^*E) = [\gamma_k(f^*\Omega)] \in H^{2k}_{dR}(N;\mathbb{C}) \qquad E \to E$$

$$f^*(c_k(E)) = f^*[\underbrace{\gamma_k(\Omega\otimes A)}_{\in H^{2k}_{dR}(M;\mathbb{C})}] = f^*[(\Lambda^k\Omega)\cdot \underbrace{\gamma_k(\overset{\Gamma(\mathrm{End}(E))}{A}(\overset{M}{x}))}_{\text{function}}]$$

$$= [f^*(\Lambda^k\Omega)\ f^*(\gamma_k(A(x)))]$$

pull-backs distribute over Λ

$$= [(\Lambda^k f^*\Omega)\cdot\gamma_k(f^*A)] = [\gamma_k(f^*\Omega\otimes f^*A)]$$

$$= [\gamma_k(f^*(\Omega\otimes A))] = c_k(f^*E)$$

$\boxed{\text{Axiom 3:}}$ Whitney Sum: $E_1, \dots, E_r$ line bdls over M

$$c_k(E_1 \oplus \dots \oplus E_r) = c_k(E_1) \cdots c_k(E_r)$$

Let $\nabla_1, \dots, \nabla_r :=$ connections on $E_1, \dots, E_r$ & $R_{\nabla_1}, \dots, R_{\nabla_r}$ be their curvature, i.e. Locally $R_{\nabla_i} s_i = \Omega_i s_i$

frame field on $\pi_i^{-1}(U) \cong \mathbb{C} \times U$

1×1 matrix of 2-forms $\Lambda^2 T^* M$

Notice $\nabla := \nabla_1 \oplus \dots \oplus \nabla_r$ is a connection on $E_1 \oplus \dots \oplus E_r$ as a rank r bundle over M.

Pick local frame field $s = (s_1, \dots, s_r)$ from above

$$R_\nabla s_j = \Omega_j s_j \implies \text{Locally } \Omega = \begin{pmatrix} \Omega_1 & & 0 \\ & \Omega_2 & \\ 0 & & \ddots \ \Omega_r \end{pmatrix}$$

$$\therefore \underbrace{\det\left(I_r - \frac{1}{2\pi i}\Omega\right)}_{c(E_1 \oplus \dots \oplus E_r)} = \underbrace{\left(1 - \frac{1}{2\pi i}\Omega_1\right)}_{c(E_1)} \wedge \underbrace{\left(1 - \frac{1}{2\pi i}\Omega_2\right)}_{c(E_2)} \wedge \dots \wedge \underbrace{\left(1 - \frac{1}{2\pi i}\Omega_r\right)}_{c(E_r)}$$

$$c(E_1 \oplus \dots \oplus E_r) = c(E_1) \cdot c(E_2) \cdots c(E_r)$$

Axiom 4: Normalization: $\int_{\mathbb{CP}^1} \gamma_1(R_\nabla) = -1$, where R_∇ is curvature of the tautological line bdl L over $\mathbb{CP}^1$

i.e. $[\gamma_1(R_\nabla)] \in H^2_{dR}(\mathbb{CP}^1)$ corresponds to $-1 \in H^2(\mathbb{CP}^1; \mathbb{Z})$

*Note $\mathbb{CP}^1 = \{pt\} \sqcup_{\varphi} D^2$, where $\varphi(x) \sim \varphi(-x)$ for $x \in \partial D^2$

Consider the natural Hermitian structure on the tautological line bdl L over $\mathbb{CP}^1$; fibers, $\pi^{-1}([x_0 : x_1]) \cong \{\text{pts in } \mathbb{C}^2 \text{ in } [x_0 : x_1]\}$

$$\hat{\mathbb{CP}}^1$$

$$h([x_0 : x_1], [x_0 : x_1]) = |x_0|^2 + |x_1|^2$$

Consider the coords: $z = x_1/x_0$ on $\mathcal{U} := \mathbb{CP}^1 \setminus \{[0:1]\}$

Take $s = (1, z)$ the frame field of L over $\mathcal{U}$.

$$\implies h(s(z), s(z)) = 1 + |z|^2$$

$\therefore$ Connection $\omega = \left(\frac{d}{dz}[h]\right) dz \cdot \frac{1}{h}$

$$= \frac{\bar{z}\, dz}{1 + |z|^2}$$

$$\Omega = d\omega = \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}\right) \wedge dz$$

$$= \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz = \left(\frac{d}{d\bar{z}}\left[\frac{\bar{z}}{1 + z\bar{z}}\right]\right) d\bar{z} \wedge dz$$

$$= \frac{1}{(1+|z|^2)^2} d\bar{z} \wedge dz$$

curvature $\Omega = d\omega + \omega \wedge \omega = \dfrac{-dz \wedge d\bar{z}}{(1 + |z|^2)^2}$

$$\implies c_1(\Omega) = \frac{1}{2\pi i} \frac{-dz \wedge d\bar{z}}{(1+|z|^2)^2}$$

$$= \frac{-2r\, dr \wedge dt}{(1+r^2)^2}$$

$\boxed{\text{change of coords } z = re^{2\pi i t}}$

$u = 1 + r^2$
$du = 2r\,dr$

$$\therefore \int_{\mathbb{CP}^1} c_1(\Omega) = \int_{\mathcal{U}} c_1(\Omega) = -\int_0^1 \int_0^\infty \frac{2r\,dr}{(1+r^2)^2} dt$$

$$= -\int_0^1 dt \int_1^\infty \frac{du}{u^2} du = \left[\frac{1}{u}\right]_1^\infty = -1$$

$\square$

Revisiting Axiom 1:

Milnor+Stasheff: **Step 1:** Use Gauss-Bonet Thm: $\int_M \Omega_{1,2} = \int_M K\,dA = 2\pi e[M]$

M = Riemannian 2-Mf'ld

+pf: Comes down to computation on a complex line bdl over M w/ a connection which plays well w/ the natural metric (from Riemannian stbre)

$\rightsquigarrow$ $[i\Omega_{1,2}] \underset{\uparrow}{=} a\cdot e(T^*M)$

since only characteristic class in $H^2(M;\mathbb{C})$ for line bdls are multiples of $e(T^*M)$

Pick $M = S^2 \Rightarrow \int i\Omega_{1,2} = i\underline{\int_{S^2} K\,dA} = \underline{2}\pi i\, e\underline{[M]}$

∴ On Line bdls $i\gamma_1(R_\nabla) = a\cdot c_1(E) = a\, e(E)$

$a = 2\pi i$ as above.

$\Rightarrow c_1(E) = \gamma_1(R_\nabla)/2\pi$

Step 2: Let E be a complex vect. bdl w/ a Connection ∇.

Case 1: E is a whitney sum of line bdls

$E = \xi_1 \oplus \cdots \oplus \xi_n$, ξ_i are line bdls.

Choose a connection on each line bdl. ∇_i

Take $\tilde{\nabla} = \bigoplus_{i=1}^{n} \nabla_i \Rightarrow$ Locally $\Omega = \begin{pmatrix} \Omega_1 & & & \\ & \Omega_2 & & \\ & & \ddots & \\ & & & \Omega_n \end{pmatrix}$

$$\therefore [\gamma(\Omega)] = \sum_{k=0}^{r} \gamma_k(\Omega) = \det(I_r - \tfrac{1}{2\pi i}\Omega) = \det(1 - \tfrac{1}{2\pi i}\Omega_1) \wedge \dots \wedge \det(1 - \tfrac{1}{2\pi i}\Omega_n)$$

$$= (1+\gamma_0(\Omega_1)) \wedge \dots \wedge (1+\gamma_0(\Omega_n))$$

$\rightsquigarrow$ By step 1 $\Rightarrow$ $= c(\xi_1)\cdots c(\xi_n)$ $\leftarrow$ axiomatic Chern

$\Rightarrow$ $= \underline{c(E)}$ $\longleftarrow$ whitney sum for axiomatic chern

$\Rightarrow$ $\therefore$ For whitney sums $c(E) = [\gamma(\Omega)]$

Step 3: Generalizing to any $E :=$ rank r vect. bdl over a compact base

$\exists$ a bdl map $E \longrightarrow \gamma^r(\mathbb{R}^{r+k})$ provided k is suff. large.

*Recall $\gamma^r(\mathbb{C}^{r+k})$ is the canonical rank r bdl over $Gr(r, \mathbb{C}^{r+k})$, w/ fibers of an r-plane in $\mathbb{C}^{r+k}$ are vectors in that r-plane

(i) $\gamma^r(\mathbb{C}^{r+k})$ is locally trivial

Take to $Gr(1,\infty)$ world:

Let γ^1 be universal line bdl over $\mathbb{C}P^k \subset Gr(1,\infty)$ w/ k large enough, then E is the pullback of γ^1.

($\exists$ bdl morphism $f: E \longrightarrow \gamma^1$)

Step 2 $\Rightarrow$ $c(\underbrace{\gamma^1 \oplus \dots \oplus \gamma^1}_{r\text{-copies}}) = c(\gamma^1)\cdots c(\gamma^1)$

γ^r is univ. bdl over $Gr(r,\infty)$;

Since $H^*(Gr(r,\infty)) \longrightarrow H^*(\mathbb{C}P^k \oplus \dots \oplus \mathbb{C}P^k)$ monomorphically in dim $\leq 2k$

$E = \overset{\text{rank } r}{\mathbb{C}\text{ vect. bdl}}$ $\Rightarrow$ it's the pullback of some bdl map: $f: E \longrightarrow \gamma^r$

Naturality:

$$\Rightarrow c(E) = c(f^*\gamma^r) = f^*c(\gamma^r) = f^*c(\bigoplus_{i=1}^{n} \gamma^1) = f^*[\gamma(R_{\oplus_{i=1}\gamma^1})] \in H^*(M;\mathbb{Z})$$

By naturality $c(f^*\gamma') = f^* c(\gamma') \in H^*(M;\mathbb{Z})$