Splicing braid varieties

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November 8, 2026

About me

I'm a professor at University of California, Davis, interested in algebraic geometry, representation theory and combinatorics, with some connections to knot invariants.




Flags and relative position

A **complete flag** in $\mathbb{C}^n$ is a collection of subspaces

$$\mathcal{F} = \{0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathbb{C}^n\}, \ \dim \mathcal{F}_i = i.$$

For example, for some choice of basis we have the standard flag

$$\mathcal{F}^{\mathrm{std}} = \{0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle = \mathbb{C}^n\}$$

and the antistandard flag

$$\mathcal{F}^{\mathrm{ant}} = \{0 \subset \langle e_n \rangle \subset \langle e_{n-1}, e_n \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle = \mathbb{C}^n\}.$$

More generally, for a permutation $w \in S_n$ we have the coordinate flag

$$\mathcal{F}(w) = \left\{0 \subset \langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots \subset \langle e_{w(1)}, \ldots, e_{w(n)} \rangle = \mathbb{C}^n\right\}.$$

Note that $\mathcal{F}^{\mathrm{std}} = \mathcal{F}(e)$ and $\mathcal{F}^{\mathrm{ant}} = \mathcal{F}(w_0)$.

Flags and relative position

Two flags $\mathcal{F}, \mathcal{F}'$ are in relative position s_i if $\mathcal{F}_j = \mathcal{F}'_j$ for $j \neq i$ and $\mathcal{F}_i \neq \mathcal{F}'_i$. We denote this by $\mathcal{F} \xrightarrow{s_i} \mathcal{F}'$. Given a reduced expression $u = s_{i_1} \cdots s_{i_k} \in S_n$, we say that two flags $\mathcal{F}, \mathcal{F}'$ are in relative position u if there exists a chain of flags

$$\mathcal{F} \xrightarrow{s_{i_1}} \mathcal{F}^{(1)} \xrightarrow{s_{i_2}} \mathcal{F}^{(2)} \ldots \xrightarrow{s_{i_k}} \mathcal{F}^{(k)} = \mathcal{F}'.$$

We will abbreviate this as $\mathcal{F} \xrightarrow{u} \mathcal{F}'$.

- If such a chain of flags exists, it is unique and $\mathcal{F}^{(i)}$ are determined by $\mathcal{F}$ and $\mathcal{F}'$.
- The definition does not depend on the choice of a reduced expression for u. We have $\mathcal{F} \xrightarrow{u} \mathcal{F}'$ if and only if

$$\dim(\mathcal{F}_i \cap \mathcal{F}'_j) = \sharp\left(\{1, \ldots, i\} \cap \{u(1), \ldots, u(j)\}\right) \text{ for all } i, j.$$

Flags and relative position

- The set of flags $\mathcal{F}'$ such that $\mathcal{F}^{\mathrm{std}} \xrightarrow{u} \mathcal{F}'$ is precisely the Schubert cell for u.
- The group $GL(n)$ acts on $\mathbb{C}^n$ and on the flag variety, preserving the relative position of flags.
- We have $\mathcal{F} \to u\mathcal{F}'$ if and only if there exists a matrix $g \in GL(n)$ such that $g\mathcal{F} = \mathcal{F}^{\mathrm{std}}$ and $g\mathcal{F}' = \mathcal{F}(u)$.

Flags and relative position

For example, $\mathcal{F} \xrightarrow{w_0} \mathcal{F}'$ if and only if $\mathcal{F}$ and $\mathcal{F}'$ are transverse: $\mathcal{F}_i \oplus \mathcal{F}'_{n-i} = \mathbb{C}^n$.

- If $\mathcal{F}$ and $\mathcal{F}'$ are transverse then there is a unique (up to scalars) basis compatible with both $\mathcal{F}, \mathcal{F}'$. Namely, for $v_i = \mathcal{F}_i \cap \mathcal{F}'_{n-i+1}$ we have $\mathcal{F}_i = \langle v_1, \ldots, v_i \rangle$ and $\mathcal{F}'_i = \langle v_{n-i+1}, \ldots, v_n \rangle$.
- If $\mathcal{F}$ and $\mathcal{F}'$ are transverse then there is a unique (up to a diagonal matrix) matrix $g \in GL(n)$ such that $g\mathcal{F} = \mathcal{F}^{\mathrm{std}}$ and $g\mathcal{F}' = \mathcal{F}^{\mathrm{ant}}$.
- Suppose that $\mathcal{F}_i = \langle m_1, \ldots, m_i \rangle$ and M is a matrix with columns m_i. Then $\mathcal{F}$ is transverse to $\mathcal{F}(w)$ if and only if

$$\mathcal{F} \pitchfork \mathcal{F}(w) \Leftrightarrow M = ww_0LU \Leftrightarrow \Delta_{ww_0[i],[i]}(M) \neq 0 \text{ forall } i = 1, \ldots, n.$$

Here L is lower-triangular and U is upper-triangular, and the decomposition is unique if we require that L has 1's on diagonal.

Braid varieties

Let $\beta = s_{i_1} \cdots s_{i_k}$ be a positive braid word, typically non-reduced. The **braid variety** $X(\beta)$ is defined as a space of sequences of flags

$$\mathcal{F} \xrightarrow{s_{i_1}} \mathcal{F}^{(1)} \xrightarrow{s_{i_2}} \mathcal{F}^{(2)} \ldots \xrightarrow{s_{i_k}} \mathcal{F}^{(k)} = \mathcal{F}'.$$

such that $\mathcal{F} = \mathcal{F}^{\mathrm{std}}$ and $\mathcal{F}' = \mathcal{F}^{\mathrm{ant}}$. Equivalently, one can require that $\mathcal{F}$ is transverse to $\mathcal{F}'$ and take the quotient by the action of $GL(n)$.

Braid varieties

- The braid variety $X(\beta)$ is non-empty if and only the Demazure product of β equals w_0, equivalently β contains w_0 as a (non-consecutive) subword.
- If non-empty, $X(\beta)$ is a smooth affine variety of dimension $\dim X(\beta) = \ell(\beta) - \ell(w_0)$.
- The braid relations $s_i s_{i+1} s_i \leftrightarrow s_{i+1} s_i s_{i+1}$ and $s_i s_j \leftrightarrow s_j s_i$ $(|i-j| > 1)$ correspond to isomorphisms of braid varieties.

Braid varieties

For $v \leq w$ we can consider **open Richardson variety** $R(v, w)$ defined as the space of flags $\mathcal{F}$ such that

$$\mathcal{F}^{\mathrm{std}} \xrightarrow{w} \mathcal{F} \xrightarrow{v^{-1}w_0} \mathcal{F}^{\mathrm{ant}}.$$

It is the intersection of the Schubert cell for w and the opposite Schubert cell for v. If we choose reduced expressions $\underline{w}$ and $\underline{v^{-1}w_0}$ for w and $v^{-1}w_0$ respectively, we can identify $R(v, w)$ with the braid variety

$$R(v, w) \simeq X\left(\underline{w} \cdot \underline{v^{-1}w_0}\right).$$

Braid varieties

The **double Bott-Samelson variety** $\mathrm{BS}(\beta)$ is defined as the space of sequences of flags

$$\mathcal{F} \xrightarrow{s_{i_1}} \mathcal{F}^{(1)} \xrightarrow{s_{i_2}} \mathcal{F}^{(2)} \ldots \xrightarrow{s_{i_k}} \mathcal{F}^{(k)} = \mathcal{F}'.$$

such that $\mathcal{F} = \mathcal{F}^{\mathrm{std}}$ and $\mathcal{F}' \pitchfork \mathcal{F}^{\mathrm{ant}}$. We have isomorphisms

$$\mathrm{BS}(\beta) \simeq X(\beta\Delta) \simeq X(\Delta\beta)$$

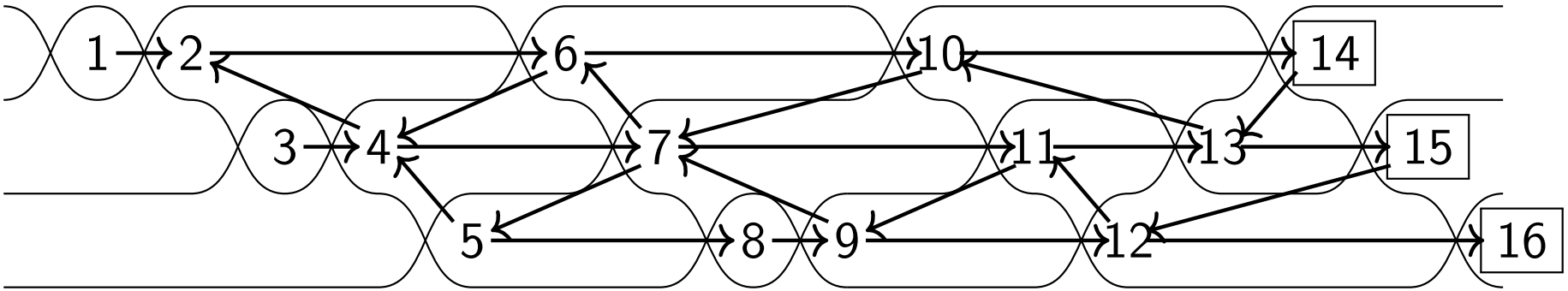
where $\Delta = \underline{w_0}$ is the half-twist braid.

Braid varieties

Theorem (Shen-Weng)

The double Bott-Samelson variety $\mathrm{BS}(\beta)$ *has a cluster structure. The cluster variables correspond to the regions of the braid, except for the left boundary.*

For $\beta = \sigma_3^2\sigma_2^2\sigma_1\sigma_3\sigma_2\sigma_1^2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$ we get



Braid varieties

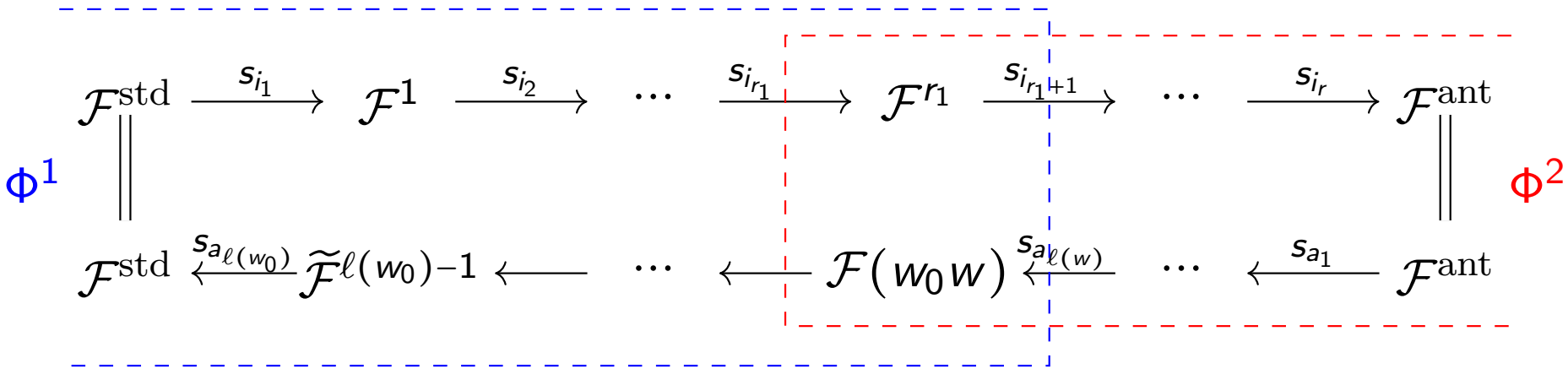
Theorem (Casals-G.-Gorsky-Le-Shen-Simental, Galashin-Lam-Sherman-Bennett-Speyer)

For any β the braid variety $X(\beta)$ has a cluster structure.

The combinatorics of the quiver for general β is significantly more complicated, and uses either algebraic weaves or 3D plabic graphs.

Splicing

Consider the following diagram:



On the top we have a sequence of flags corresponding to some braid β which we factor as $\beta^1\beta^2$. On the bottom we have a sequence of coordinate flags corresponding to a reduced word for w_0. If we require that the flag $\mathcal{F}^{r_1}$ is transverse to the coordinate flag $\mathcal{F}(w_0 w)$, the diagram splits into two.

Splicing

More formally, we can define an open subset $\mathcal{U}_{r_1,w} \subset X(\beta)$ where $\mathcal{F}^{r_1}$ is transverse to the coordinate flag $\mathcal{F}(w_0 w)$. It is given by non-vanishing of explicit flag minors for $\mathcal{F}^{r_1}$. Then we have the following:

Theorem (G.-Kim-Scroggin-Simental)

For each $r_1 = 1, \ldots, \ell(\beta)$ and $w \in S_n$, we have an isomorphism of algebraic varieties

$$\Psi_{r_1,w} : X\left(\underline{(w^{-1}w_0)}\beta^1\right) \times X\left(\beta^2 \underline{w}\right) \xrightarrow{\cong} \mathcal{U}_{r_1,w}.$$

Note that $\mathcal{U}_{r_1,w}$ could be empty if one of the braid varieties on the left hand side is empty.

Splicing

For open Richardson varieties, we get the following corollary.

Corollary

For $v \leq u \leq w$, we have an open embedding

$$R(v,u) \times R(u,w) \hookrightarrow R(v,w).$$

Its image is the principal open subset given by non-vanishing of flag minors $\Delta_{u[i],[i]}$, $i = 1, \ldots, n$.

Corollary

Suppose that $v = v_0 < v_1 < \ldots < v_k = w$ is a maximal chain from v to w in the Bruhat order. Then we have an open embedding

$$(\mathbb{C}^\times)^k = R(v_0,v_1) \times R(v_1,v_2) \cdots \times R(v_{k-1},v_k) \hookrightarrow R(v,w).$$

Splicing

Corollary

For double Bott-Samelson varieties, we have an open embedding

$$\mathrm{BS}(\beta^1) \times \mathrm{BS}(\beta^2) \hookrightarrow \mathrm{BS}(\beta)$$

Its image is the open subset $\mathcal{U}_{r_1}$ defined by non-vanishing of some explicit minors.

Splicing

We conjecture that the splicing map behaves well with respect to the cluster structures.

Conjecture (GKSS)

For each $r_1 = 1, \ldots, r$ and $w \in S_n$ such that $\mathcal{U}_{r_1,w} \neq \emptyset$, there exists a seed $\Sigma = (Q, \mathbf{x})$ in $\mathbb{C}[X(\beta)]$ with cluster variables $x_{a_1}, \ldots, x_{a_s} \in \mathbf{x}$ such that:

(a) *The open set $\mathcal{U}_{r_1,w}$ is the common non-vanishing locus of $x_{a_1}, \ldots, x_{a_s}$.*

(b) *The variety $\mathcal{U}_{r_1,w}$ is the cluster variety associated to the seed obtained from Σ upon freezing the cluster variables $x_{a_1}, \ldots, x_{a_s}$.*

(c) *The splicing map is a cluster quasi-isomorphism.*

In particular, conjectures (a) and (b) mean that the minors defining $\mathcal{U}_{r_1,w}$ are cluster monomials in Σ, and $x_{a_1}, \cdots, x_{a_s}$ are their irreducible factors.

Splicing

For Richardson varieties, the conjecture would imply that a maximal chain $v = v_0 < v_1 < \ldots < v_k = w$ (or, more generally, a maximal chain with ordering of its elements) defines a cluster torus

$$(\mathbb{C}^\times)^k = R(v_0, v_1) \times R(v_1, v_2) \cdots \times R(v_{k-1}, v_k) \hookrightarrow R(v, w).$$

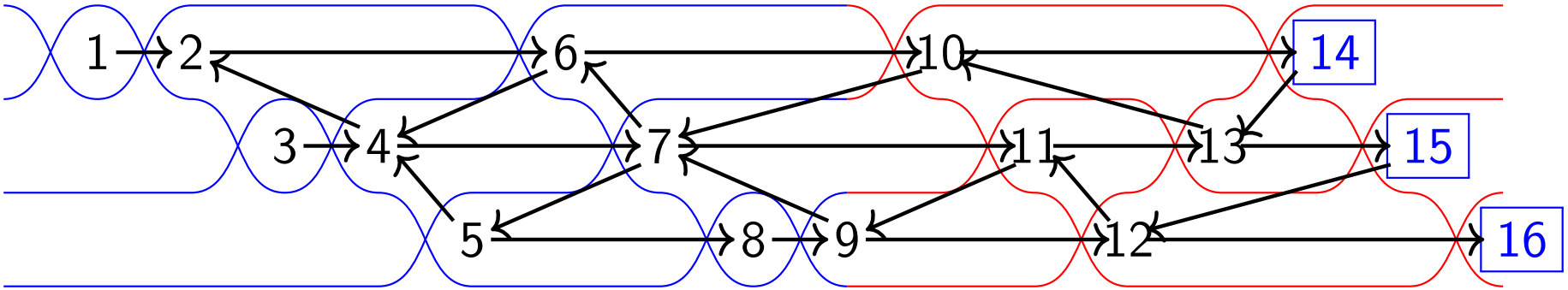
In would be very interesting to classify all such tori.

Splicing

Theorem (GKSS)

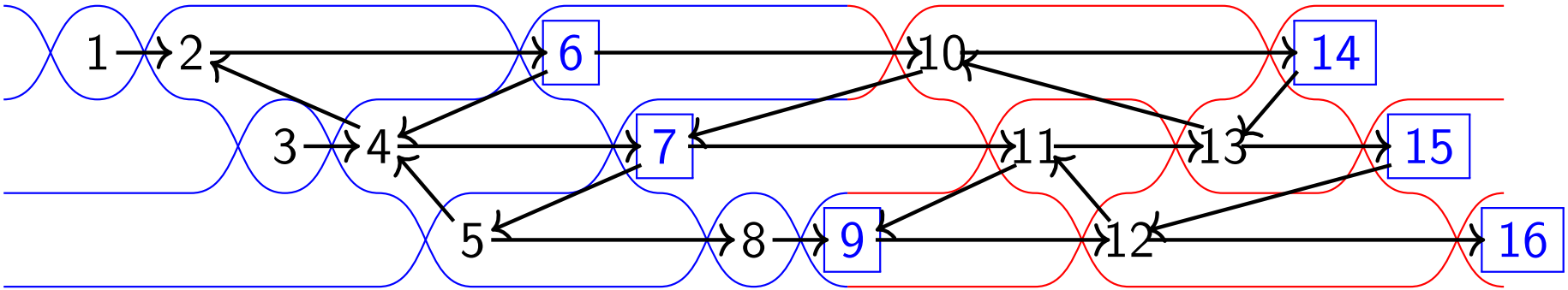
The conjecture holds for double Bott-Samelson varieties.

Consider $\beta = \sigma_3^2\sigma_2^2\sigma_1\sigma_3\sigma_2\sigma_1^2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$, where β^1 and β^2 are as indicated by colors. We have the quiver Q_β:



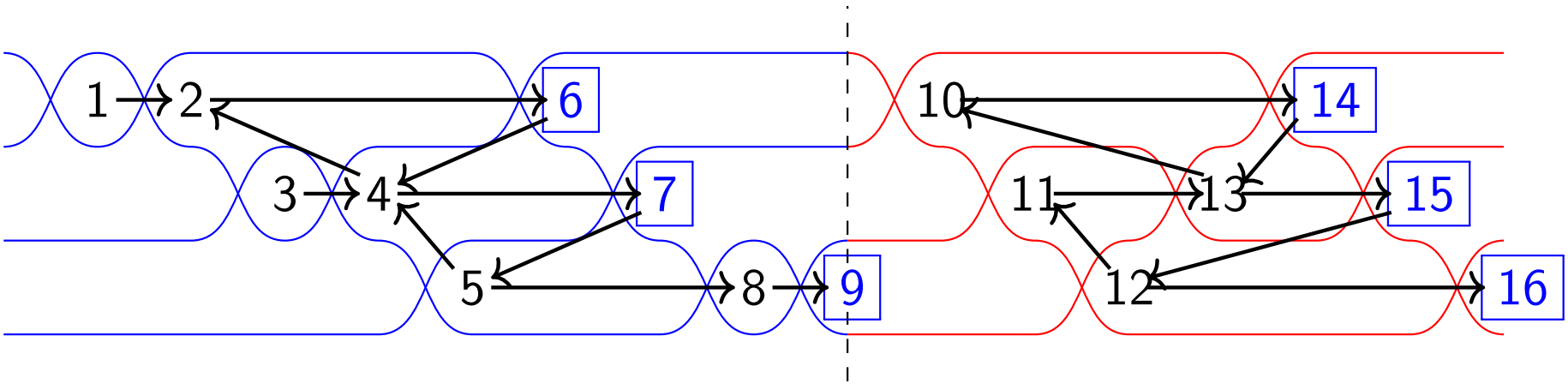
Splicing

The open set $\mathcal{U}_{r_1}$ is the cluster variety corresponding to the following quiver, which is obtained by freezing the vertices $6, 7, 9$. Note that these are the vertices on the right of the rightmost appearance of a crossing in β^1.



Splicing

On the other hand, the product $\mathrm{BS}(\beta^1) \times \mathrm{BS}(\beta^2)$ is the cluster variety corresponding to the disjoint union of the quivers Q_{β^1} and Q_{β^2}:



Splicing

For general braid varieties, we can prove parts of the conjecture when $w = e$ or $w = w_0$.

Yuma Mizuno recently proved the conjecture for the special case of splicing map

$$X(\beta) \times X(\Delta\Delta) \hookrightarrow X(\beta\Delta).$$

Thank you!