Operations in link homology

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October 29, 2026

A link homology theory $\mathcal{H}$ assigns a vector space to each link diagram L, such that $\mathcal{H}(L) \simeq \mathcal{H}(L')$ for equivalent link diagrams $L \sim L'$. More generally, given a link cobordism $\Sigma : L \to L'$, one often gets a linear map $F_\Sigma : \mathcal{H}(L) \to \mathcal{H}(L')$. In this talk we will focus on Khovanov and Khovanov-Rozansky homology and their cousins.

An **operation** A is an assigment of linear maps $A : \mathcal{H}(L) \to \mathcal{H}(L)$ for each link diagram L, such that the isomorphisms $\mathcal{H}(L) \simeq \mathcal{H}(L')$ can be chosen to be compatible with A.

One can also ask if A is compatible with cobordism maps F_Σ for some class of cobordisms Σ.

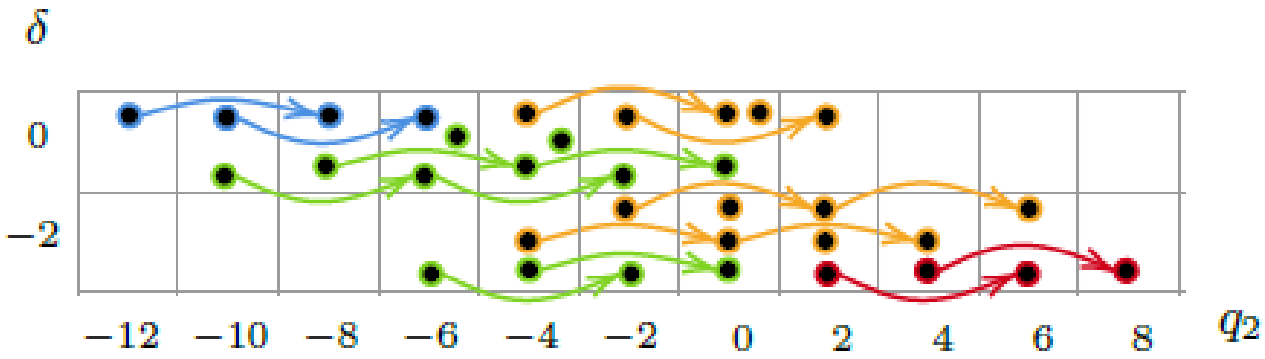
Why do we need operations?

1. To distingush links: if there is no isomorphism between $\mathcal{H}(L)$ and $\mathcal{H}(L')$ preserving gradings and operations, then L and L' are not equivalent.
2. To name some interesting classes in homology: often $\mathcal{H}(L)$ is one-dimensional in some specific degree, this gives a distinguished class $x \in \mathcal{H}(L)$ (defined up to a nonzero scalar). One can then consider more complicated classes $A(x)$ for various operations A.
3. To uncover new structures and symmetries of $\mathcal{H}(L)$.
4. To constrain the cobordism maps $F_\Sigma : \mathcal{H}(L) \to \mathcal{H}(L')$.

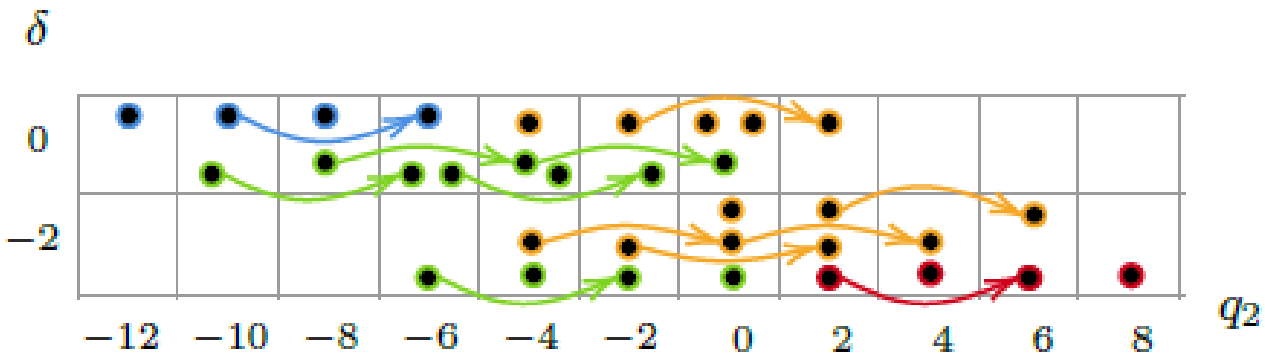
In this talk, I will give examples of all these applications.

Theorem (T. Sano)

a) There is an operation $e : \mathrm{Kh}(L) \to Kh(L)$ *in Khovanov homology which preserves* δ*-grading and changes* q*-grading by* 4*.*
b) While Khovanov homologies of the Conway knot and of the Kinoshita-Terasaka knot are isomorphic, the actions of e *are distinct!*



(b) $\widetilde{Kh}(K_C^*)$



(c) $\widetilde{Kh}(K_{KT}^*)$

Figure 3: Distinguishing K_C and K_{KT} by the action of e

[1] Picture credit: Taketo Sano, arXiv:2602.17435

Let $\overline{\mathrm{HHH}}(K)$ denote the **reduced** triply graded HOMFLY homology of the knot K.

Theorem (G.,Hogancamp,Mellit)

a) For any knot K there is an action of the family of commuting operators $F_k, k \geq 2$ in $\overline{\mathrm{HHH}}(K)$.
b) The action of F_2 can be extended to the action of the Lie algebra

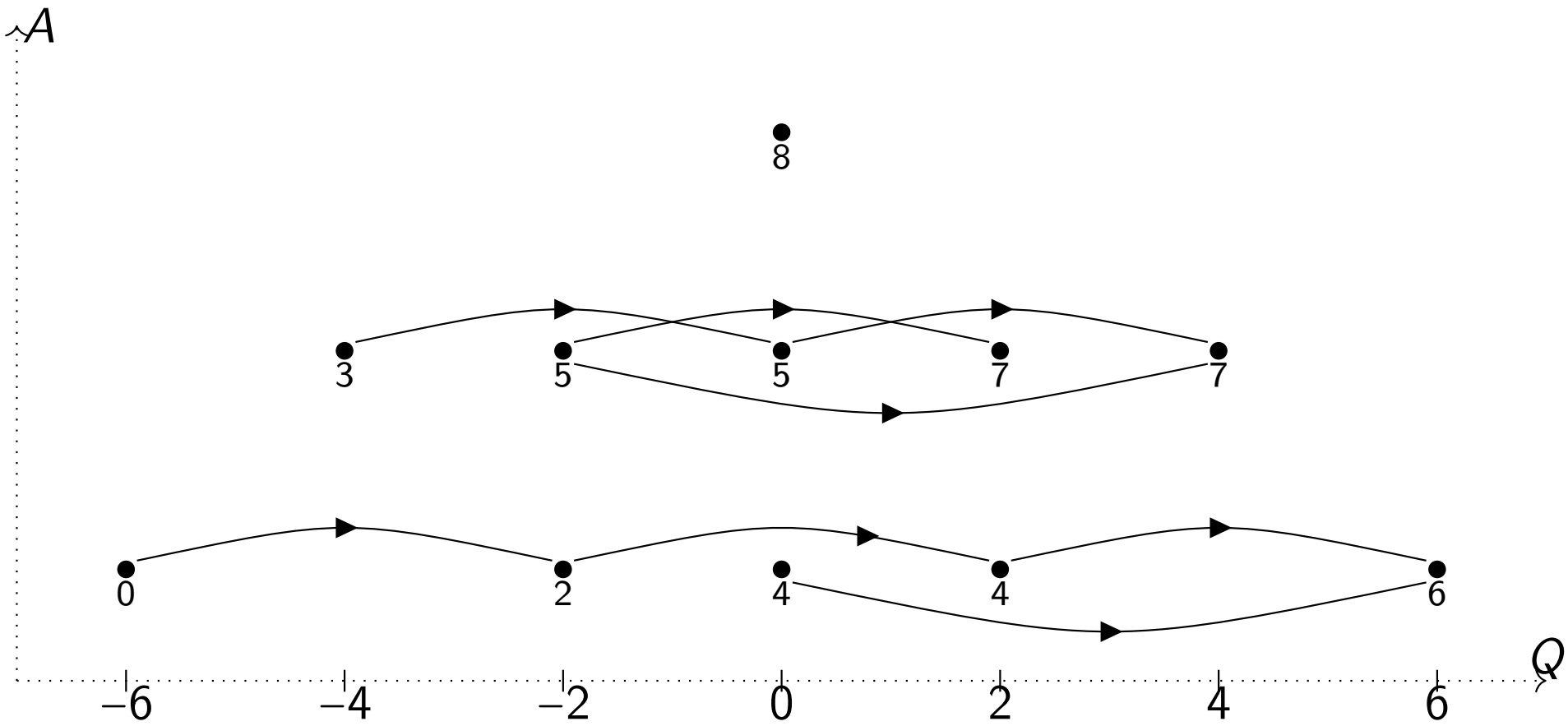
$$\mathfrak{sl}_2 = \langle e, f, h \rangle \mapsto \langle E_2, F_2, h \rangle$$

where h acts by multiplication by the q-degree.

The last statement can be rephrased using $\mathfrak{sl}_2$ representation theory. If $V = \bigoplus V_j$ is a finite-dimensional representation of $\mathfrak{sl}_2$ and V_j are weight spaces then $f^j : V_j \to V_{-j}$ is an isomorphism for all $j \geq 0$, this is known as the **Lefshetz property**. Conversely, if F_2 satisfies Lefshetz property then there exists a unique operator E_2 extending it to an $\mathfrak{sl}_2$-triple.

Corollary

The reduced HOMFLY homology $\overline{\mathrm{HHH}}(K)$ *has a* **symmetry** *categorifying the symmetry* $P_K(q,a) = P_K(q^{-1},a)$ *of the HOMFLY polynomial.*



Operators F_k and differentials for $T(3,4)$: F_2 and F_3 correspond to shorter and longer horizontal arrows.

Theorem (Trinh)

For a positive braid β, $\mathrm{HHH}^0(\beta)$ is isomorphic to the equivariant Borel-Moore homology of the braid variety $X(\beta\Delta)$. If β closes to a knot then $X(\beta\Delta)$ has a free action of some torus T and $\overline{\mathrm{HHH}}^0(\beta)$ is isomorphic to (non-equivariant) Borel-Moore homology of $X(\beta\Delta)/T$.

Dually, we can consider cohomology of $X(\beta\Delta)$ and $X(\beta\Delta)/T$. Since these varieties are smooth and affine, the cohomology classes are represented by differential forms.

It is expected that F_k corresponds to the multiplication by some $(2k-2)$-form. In particular, F_2 corresponds to the **cluster 2-form**, and Mellit in a separate work proved that it extends to $\mathfrak{sl}_2$ action.

The operators $F_3, F_4, \ldots$ correspond to some 4-form, 6-form etc which are yet to be computed.

Problem

Describe the algebra generated by E_2 and F_k, and compute its action in $\overline{\mathrm{HHH}}(K)$.

Theorem (G.,Mellit, 2026)

For $K = T(n, n+1)$ torus knot the operators E_2 and F_k generate the action in $\overline{\mathrm{HHH}}(K)$ of the subalgebra $\mathsf{H}_{\geq 2}$ of the Lie algebra of **Hamiltonian vector fields** *on $\mathbb{C}^2$.*

More generally, we expect that the same is true for any knot.

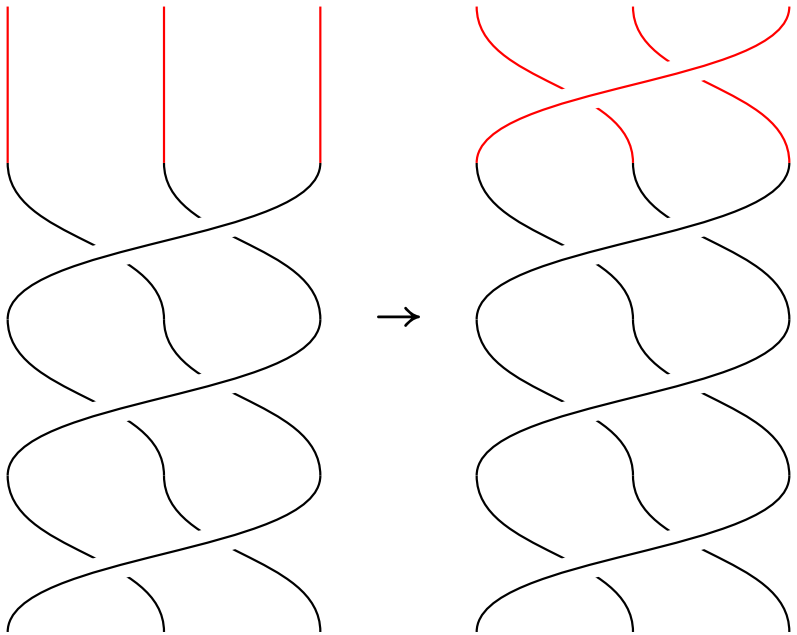
Conjecture (Rozansky, G.-Mellit)

For any knot K the operators E_2 and F_k generate the action of $\mathsf{H}_{\geq 2}$.

Here is an interesting topological application. Elias and Krasner defined maps in HHH for braid-like cobordisms between braid closures. An example of such cobordism is a saddle between the identity braid and a positive crossing.

Theorem (G., Mellit)

The natural cobordism map $\mathrm{HHH}(T(n,n)) \to \overline{\mathrm{HHH}}(T(n,n+1))$ *is surjective.*



The **Poisson bracket** on $\mathbb{C}[x, y]$ is given by

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

It is antisymmetric in f, g, derivation in both arguments:

$$\{f, gh\} = \{f, g\}h + g\{f, h\},$$

and satisfies Jacobi identity

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}.$$

For example,

$$\{x^a y^b, x^c y^d\} = (ad - bc)x^{a+c-1}y^{b+d-1}.$$

Given $f \in \mathbb{C}[x, y]$, the corresponding Hamiltonian vector field is defined as

$$X_f = \{f, -\} = \frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}.$$

The Jacobi identity implies

$$[X_f, X_g] = \left[X_{\{f,g\}}\right],$$

so the Hamiltonian vector fields form a Lie algebra H. It is spanned by $X_{a,b} = X_{x^a y^b}$ with relations

$$[X_{a,b}, X_{c,d}] = (ad - bc)X_{a+c-1,b+d-1}.$$

We define

$$\mathrm{H}_{\geq 2} = \mathrm{span}\{X_{a,b} : a + b \geq 2\}.$$

One can check that this is a Lie subalgebra of H.

For example, we have the following interesting elements in H:

1. $F_k = X_{k,0} = kx^{k-1}\frac{\partial}{\partial y}$.
2. $E_k = -X_{0,k} = ky^{k-1}\frac{\partial}{\partial x}$.
3. We have $[F_k, F_m] = 0$ and $[E_k, E_m] = 0$.
4. Up to rescaling, $F_2 = 2x\frac{\partial}{\partial y}$ and $E_2 = 2y\frac{\partial}{\partial x}$ generate a copy of $\mathfrak{sl}_2$.
5. The elements F_k, E_k for $k \geq 2$ generate $\mathrm{H}_{\geq 2}$. In fact, F_k for $k \geq 2$ and E_2 already generate $\mathrm{H}_{\geq 2}$.
6. $L_k = -X_{k,1} = x^k\frac{\partial}{\partial x} - kx^{k-1}y\frac{\partial}{\partial y}$. They generate a copy of the **Witt algebra**
$$[L_k, L_m] = (m-k)L_{k+m-1}.$$
7. Furthermore, $[F_k, E_2] = -[X_{k,0}, X_{0,2}] = -2kX_{k-1,1} = 2kL_{k-1}$.

To extend the above constructions to links and unreduced homology, we need a deformation, or y-ification. For Khovanov homology, this was defined by Batson-Seed (and expanded in the recent work of Sano), while G.-Hogancamp defined y-ified triply graded link homology $\mathrm{HY}(L)$.

For an r-component link $\mathrm{HY}(L)$ is a module over $\mathbb{C}[x_1, \ldots, x_r, y_1, \ldots, y_r]$. If K is a knot then

$$\mathrm{HY}(K) = \overline{\mathrm{HHH}}(K) \otimes \mathbb{C}[x, y].$$

so we do not get any new information.

Theorem (G.,Hogancamp,Mellit)

a) For any link L there is an action of the family of commuting operators $F_k, k \geq 1$ in $\mathrm{HY}(L)$.
b) The action of F_2 can be extended to the action of the Lie algebra

$$\mathfrak{sl}_2 = \langle e, f, h \rangle \mapsto \langle E_2, F_2, h \rangle$$

where h acts by multiplication by q-degree.
c) One has $[F_k, x_i] = 0$, $[F_k, y_i] = k x_i^{k-1}$.

As a consequence, $\mathrm{HY}(L)$ has a symmetry which exchanges the actions of x_i and y_i.

Furthermore, one can check $[E_2, x_i] = 2y_i, [E_2, y_i] = 0$.

Conjecture

The action of F_k and E_2 in $\mathrm{HY}(L)$ extends to the action of the Lie algebra of Hamiltonian vector fields H. *This action is compatible with the action of $\mathbb{C}[x_1, \ldots, x_r, y_1, \ldots, y_r]$ as follows:*

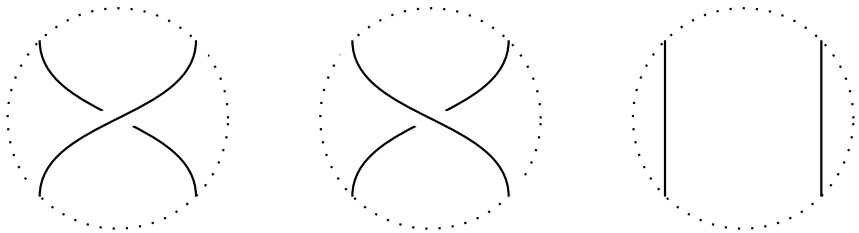
$$[X_f, x_i] = -\frac{\partial f}{\partial y_i}, \ [X_f, y_i] = \frac{\partial f}{\partial x_i}.$$

Example

For the n-component unlink we have $\mathrm{HY}(O_n) = \mathbb{C}[x_1, \ldots, x_n, y_1 \ldots, y_n, \theta_1, \ldots, \theta_n]$ and the action of X_f is given by

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial}{\partial x_i}.$$

Let L_+, L_- and L_0 be three link diagrams which agree outside of one place where we respectively put a positive crossing, a negative crossing and an oriented resolution:



There is an exact triangle

$$\mathrm{HY}(L_+) \to \mathrm{HY}(L_-) \to [\mathrm{HY}(L_0) \to \mathrm{HY}(L_0)] \to \mathrm{HY}(L_+)$$

categorifying the skein relation for the HOMFLY polynomial.

Theorem (G., Hogancamp, Mellit)

The three maps in the skein exact triangle preserve the q-degree and commute with the action of F_k for all k.

Corollary

The three maps in the skein exact triangle commute with the action of E_2.

Theorem (G., Hogancamp)

The braid-like cobordism from $T(n,n)$ to $T(n,0)$ which changes half of positive crossings to negative induces an **injective** *map*

$$\Psi : \mathrm{HY}(T(n,n)) \to \mathrm{HY}(T(n,0)) = \mathbb{C}[x_1, \ldots, x_n, y_1 \ldots, y_n, \theta_1, \ldots, \theta_n].$$

The image of Ψ is the ideal $\mathcal{J}$ generated by the antisymmetric polynomials.

Corollary

The action of E_2 and F_k on $\mathrm{HY}(T(n,n))$ commutes with Ψ and extends to the action of H.

One can also check directly that E_2 and F_k preserve the ideal $\mathcal{J}$.

Next, we would like to understand the homology of $T(n, n+1)$.

Lemma

The braid-like cobordism from $T(n,n)$ to $T(n,n+1)$ induces a map

$$\pi : \mathrm{HY}(T(n,n)) \to \mathrm{HY}(T(n,n+1)) \simeq \overline{\mathrm{HHH}}(T(n,n+1))[x,y]$$

which commutes with E_2 and F_k and factors through $\mathrm{HY}(T(n,n))/(x_i - x, y_i - y)\mathrm{HY}(T(n,n))$.

This is our main result:

Theorem

The map above induces an isomorphism

$$\mathrm{HY}(T(n,n))/(x_i, y_i)\mathrm{HY}(T(n,n)) \xrightarrow{\simeq} \overline{\mathrm{HHH}}(T(n,n+1)).$$

By the work of Hogancamp, the two spaces have the same dimension, so it is sufficient to show injectivity.

We will need the following hard result of Haiman, related to the geometry of the Hilbert scheme of points on the plane.

Theorem (Haiman)

Let $J \subset \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ be the ideal generated by antisymmetric polynomials, and

$$F_k = \sum_i k x_i^{k-1} \frac{\partial}{\partial y_i}, k \geq 2.$$

Then F_k pairwise commute and are well-defined on the quotient $J/(x_i, y_i)J$.

Furthermore, the element $\Delta = \prod_{i<j}(x_i - x_j)$ **co-generates** *$J/(x_i, y_i)J$ under the action of F_k: for any nonzero $f \in J/(x_i, y_i)J$ there exists a polynomial $p(F_2, F_3, \dots.)$ such that $p(F_2, F_3, \dots.)f = \Delta$.*

Corollary

Assume M has an action of F_k for $k \geq 2$ and $\pi : J/(x_i, y_i)J \to M$ is a linear map which commutes with F_k. If $\pi(\Delta) \neq 0$ then π is **injective**.

Topologically, we are able to identify the class in $\mathrm{HY}(T(n,n))$ corresponding to Δ and prove that $\pi(\Delta) \neq 0$ for the map

$$\pi : \mathrm{HY}(T(n,n)) \to \mathrm{HY}(T(n,n+1)).$$

This proves the main theorem in a-degree zero.

To access the higher a-degrees, we need more operators.

Theorem (Rasmussen)

There is a spectral sequence with E_1 page isomorphic to $\mathrm{HHH}(L)$ and E_∞ page isomorphic to $\mathfrak{gl}(N)$ Khovanov-Rozansky homology of L. In particular, one has a differential $d_N : \mathrm{HHH}(L) \to \mathrm{HHH}(L)$.

Theorem (Chandler-G.,Ballinger-G.-Hogancamp-Wang,Sano)

The Rasmussen spectral sequence can be y-ified, in particular, one has a differential $d_N : \mathrm{HY}(L) \to \mathrm{HY}(L)$. Furthermore. d_N commutes with the action of braid-like cobordisms and with the action of F_k for all k.

For the unlink, we have $d_N(\theta_j) = x_j^N$ and $d_N(x_j) = d_N(y_j) = 0$. This determines the action of d_N on

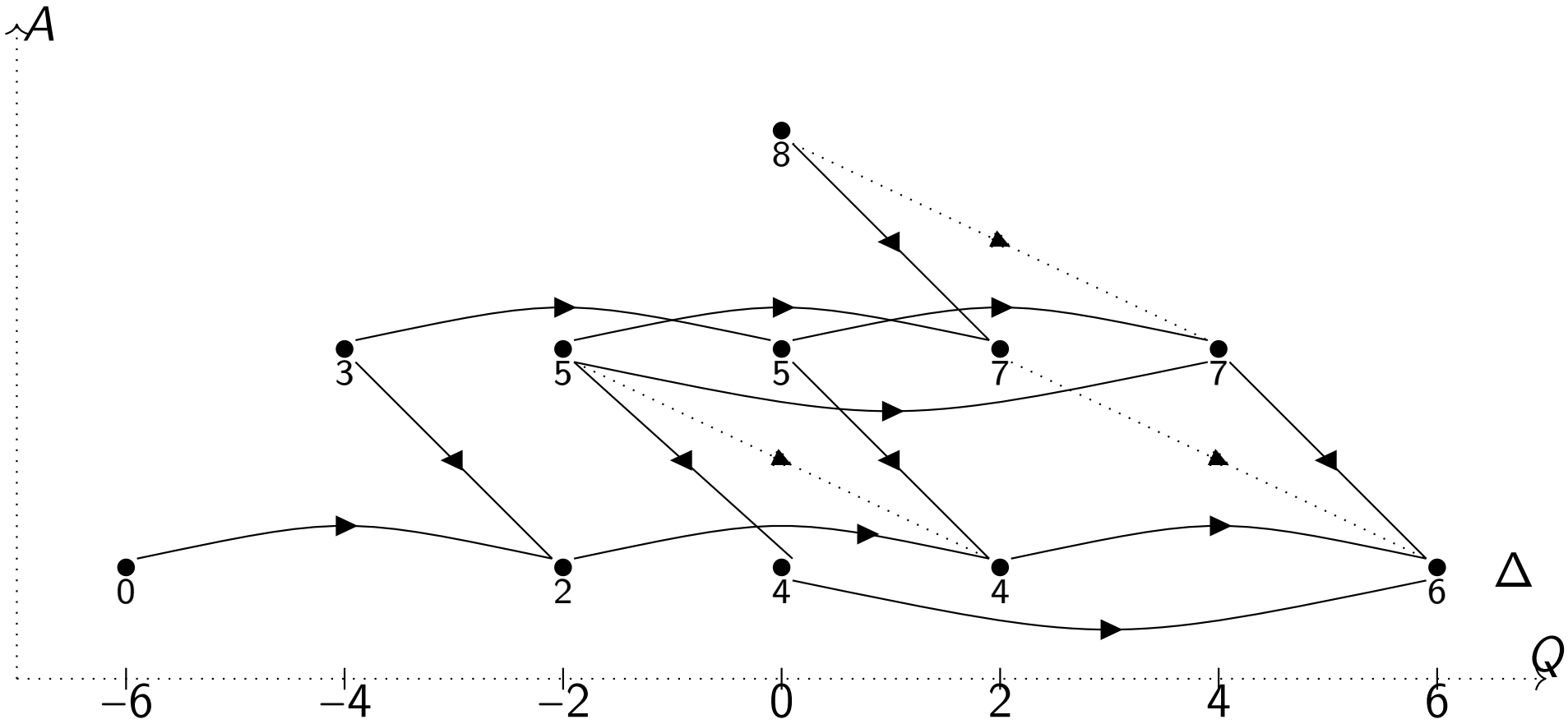
$$\mathrm{HY}(T(n,n)) \simeq \mathcal{J}$$

where $\mathcal{J}$ is the ideal in $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n, \theta_1, \ldots, \theta_n]$ generated by the antisymmetric polynomials. We have the following extension of Haiman's theorem:

Theorem (G.,Mellit)

The element $\Delta = \prod_{i<j}(x_i - x_j)$ **co-generates** $\mathcal{J}/(x_i, y_i)\mathcal{J}$ *under the action of* F_k *and* d_N.

This implies the proof of the main theorem for all a-degrees.



Operators F_k and differentials for $T(3,4)$: F_2 and F_3 correspond to shorter and longer horizontal arrows; d_1 and d_2 correspond to solid and dotted diagonal arrows.

Open problems

- Can we describe the action of E_2, F_k for more knots and links? Say, (m, n) torus knots for all m, n?
- Khovanov and Rozansky defined a Witt algebra action in $\mathrm{HHH}(L)$. How does it interact with H?
- What are the interesting operations in $\mathfrak{gl}(N)$ homology?

Thank you!