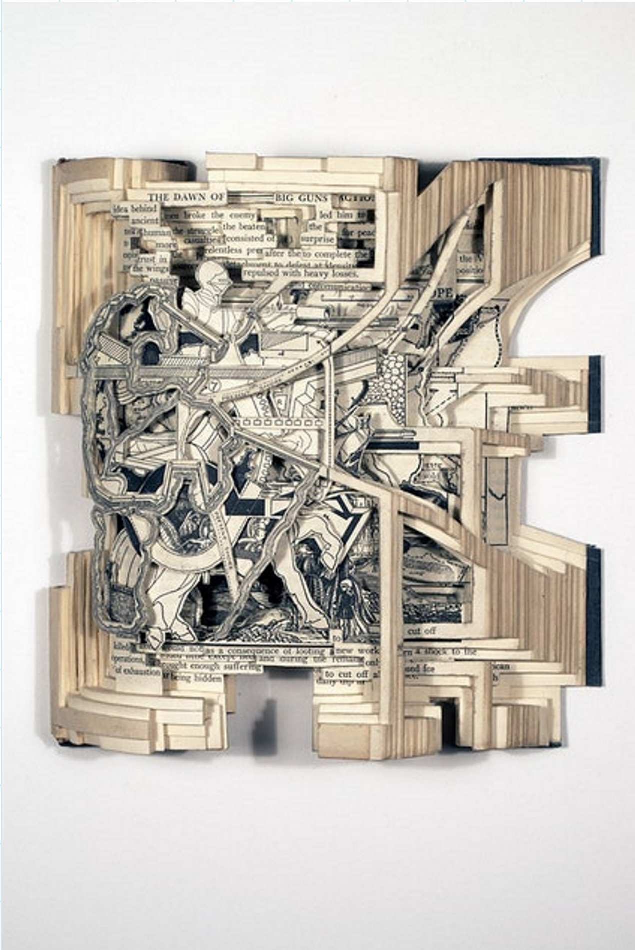
Meeting 4/7

Tuesday, April 6, 2021 4:37 PM



Brian Dettmer, "Postmodern deconstruction"

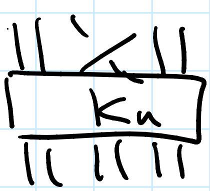
How to compute $H^*(Gr_{\mathbb{C}}(k,n))$ as a ring?

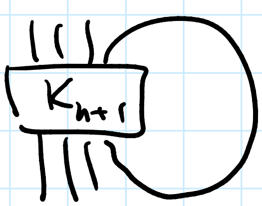
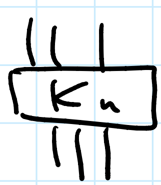
- Develop cellular homology
- Enumerate Schubert cells $\Rightarrow$ homology is even!
- Algebraic tricks (e.g. equivariant homology)

---

① <u>Thm</u> (Hogancamp) $\exists$ complex of Soergel bimodules K_n such that:

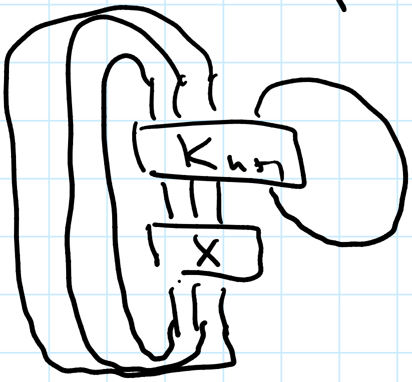
$\boxed{K_n = \text{compact categorical Jones–Wenzl projector}}$

-  $= K_n =$  (eats crossings)

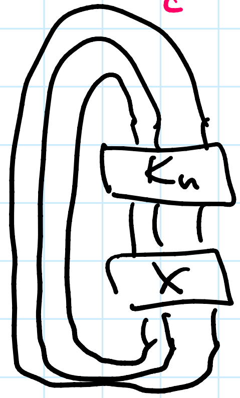
-  $= (t^n + a)$ 

braid

Abbreviation for:



$= (t^h + a)$

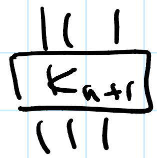


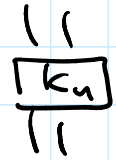
braid closure

$\Updownarrow$

HH

• 

$= t^{-h} \Big($ 

$\longrightarrow q$ | 

$\Big)$

There is a chain map here such that its cone is homotopy equivalent to the complex on the left.

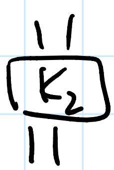
• 

$=$ |

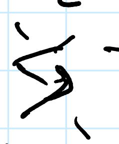
a, q, t = grading shifts

$$\left.\begin{aligned} q &= Q^2 \\ t &= T^2 Q^{-2} \\ a &= A Q^{-2} \end{aligned}\right\}$$ A, Q, T = standard HOMFLY gradings.

even homological (T) degree.

Ex: 

$= [R \longrightarrow B \longrightarrow B \longrightarrow R] = K_2$

$=$ [] $\longrightarrow$ [] $= R \longrightarrow$ []

Exercise: $K_2 \otimes T \simeq K_2 \simeq K_2 \otimes T^{-1}$

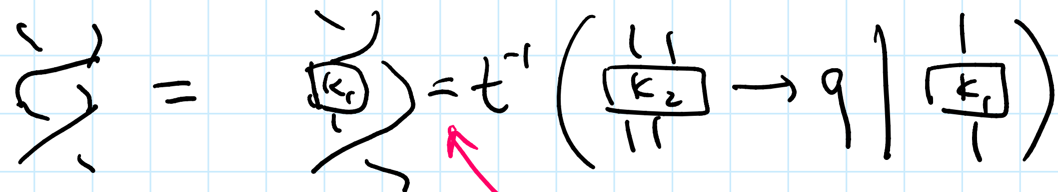
Exercise: $K_2 \otimes T \simeq K_2 \simeq K_2 \otimes T^{-1}$

where $T = [B \to R]$, $T^{-1} = [R \to B]$.

---

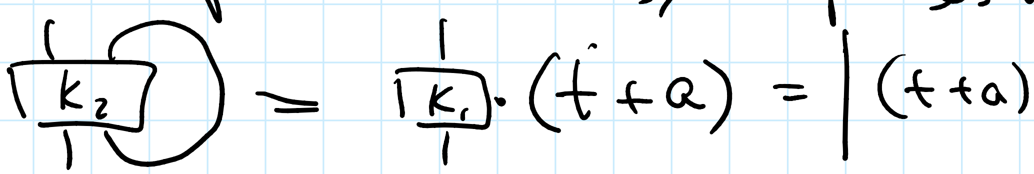
② We can use K_n to compute HHH for many links:

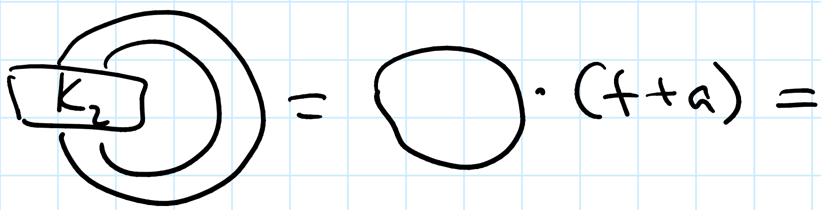
Ex:



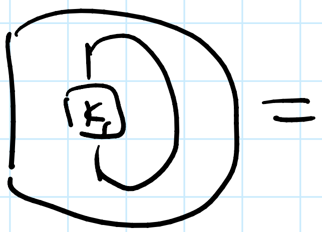
defining relation.

Close up all braids/complexes of SBim:

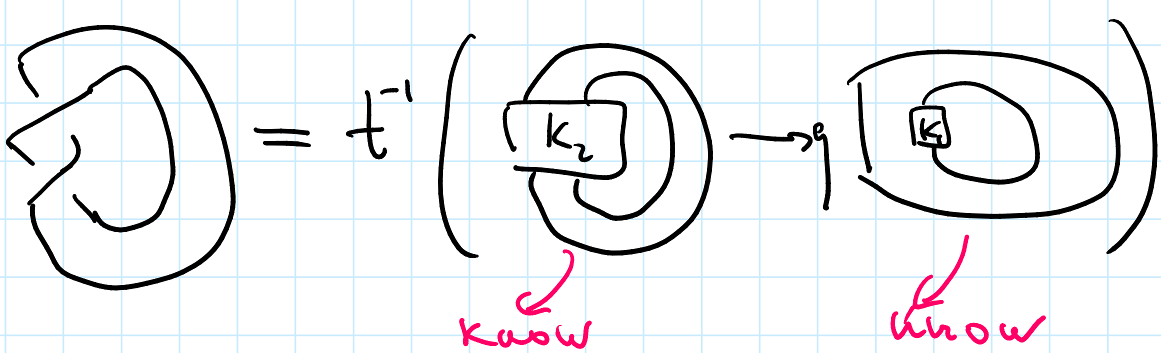




$$= \frac{(t+a)(1+a)}{1-q}.$$



$$= \frac{(1+a)^2}{(1-q)^2}.$$



know ↙ know ↙
both supported in <u>even</u>
homological degrees!

Recall, t has homological degree 2!

$\Rightarrow$ long exact sequence in homology splits!

$\Rightarrow$ to compute homology of , we just add up the terms on the right.

<u>Lemma</u>: $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$

If homology of $A^\bullet$ and $C^\bullet$ are all supported in even homological degrees, then same for $B^\bullet$, and $H^i(B) = H^i(A) \oplus H^i(C)$.

<u>Remark</u> If $A^\bullet, B^\bullet, C^\bullet$ = complexes of R-modules, more subtle: $0 \to H^i(A) \to H^i(B) \to H^i(C) \to 0$

$H^i(B)$ = extension between $H^i(A)$ and $H^i(C)$.

<u>Thm</u> (Hogancamp, Mellit) (a) $HHH(T(m,n))$ is supported in <u>even</u> homological degrees

supported in <u>even</u> homological degrees

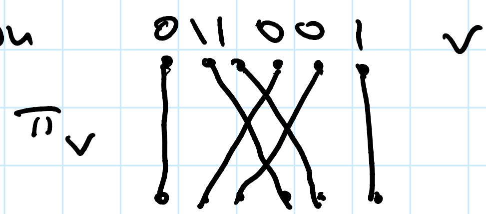
(b) Consider the following family of functions

$p(v,w)$, v,w = binary sequences $|v|=|w|=\ell$

- $p(\emptyset, 0^n) = \left(\frac{1+a}{1-q}\right)^n \quad p(0^m,\emptyset) = \left(\frac{1+a}{1-q}\right)^m$
- $p(v1, w1) = (t^\ell + a)\, p(v,w)$
- $p(v0, w1) = p(v, 1w)$, $p(v1,w0) = p(1v,w)$
- $p(v0,w0) = t^{-\ell} p(1v,1w) + q t^{-\ell} p(0v,0w)$.

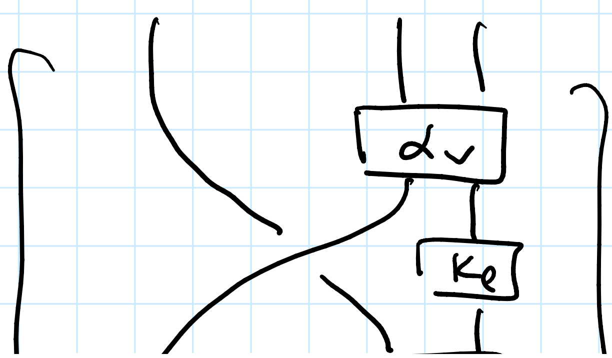
Then $HHH(T(m,n))$ is free over $\mathbb{Z}$ of triply graded rank $p(0^m,0^n)$.

<u>Idea of proof:</u> Given a binary sequence v, can draw shuffle permutation



α_v = positive braid lift of π_v

β_v = positive braid lift of π_v^{-1}



v,w = binary sequences.

$=: C(v,w)$

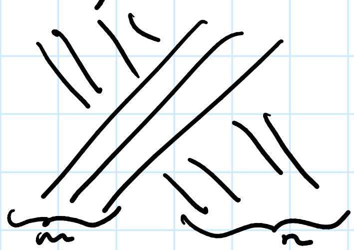
$T(m,n)$ one

m strands $\quad n \quad \ell$ strands.

$T(m,n)$ one component colored with S^ℓ $\Rightarrow$ even

<u>Claim</u>: $HHH(C(v,w)) = p(v,w)$ satisfies same recursion

$v = 0^m, \ w = 0^n \Rightarrow \ell = 0$



close this $\Rightarrow T(m,n)$ torus link.

---

③ <u>y-ification</u>

Recall: $L \longrightarrow HY(L)$ y-ified homology deformation of HHH.

<u>Thm</u> HHH is supported in even degrees

$\Rightarrow HY$ is free over $\mathbb{C}[y_1 \dots y_c]$

<u>Proof</u>: Spectral sequence $HHH(L) \otimes \mathbb{C}[y]$

$$\Downarrow$$

$$HY(L)$$

all in even degrees

$\Rightarrow$ s.s. collapses. ∎

<u>Rmk</u> X = top. space with torus action

Rmk X = top. space with torus action
$H^*(X)$ is even $\Rightarrow$ $H_T^*(X)$ is free over $H_T^*(pt)$.

Thm (G., Hogancamp) $k \geq 0$

(a) $$HY(T(n,kn)) \overset{(*)}{=} \bigcap_{i\neq j} (x_i - x_j, y_i - y_j, \theta_i - \theta_j)^k = J_k$$

ideal in $\mathbb{C}[\underbrace{x_1 \dots x_n, y_1 \dots y_n}_{\text{even}}, \underbrace{\theta_1 \dots \theta_n}_{\text{odd}}] = HY(\text{unlink})$.

$(*)$ true as an iso of $\mathbb{C}[x_1 \dots x_n, y_1 \dots y_n, \theta_1 \dots \theta_n]$-mod.

(b) $$HHH(T(n,kn)) = J_k / (y) J_k$$

Idea of proof: $HHH(T(n,kn))$ even $\Rightarrow$ $HY(T(n,kn))$ free over $\mathbb{C}[y]$.

$\rightsquigarrow$ sufficient to understand for "generic y" $y_i \neq y_j$

codim 1 $\rightsquigarrow$ only one pair $y_i = y_j$

$\Rightarrow$ reduction to $n=2$.

Rmk Very hard combinatorial problem to match this description of HHH with the above recursion.

Open questions: (a) This recursion gives a filtration on $HY/HHH(T(n,nk)) \simeq J_k$

How to think about this filtration/gr explicitly?

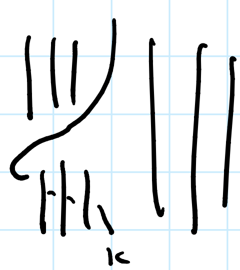
(b) Understand $HY/HHH(T(km,kn))$ $\gcd(m,n)=1$ as a module over $\mathbb{C}[x_1 \ldots x_k, y_1 \ldots y_k, \theta_1 \ldots \theta_k]$?

(c) For which other links the recursion works? When HHH is even?

(d) Over-optimistic conjecture:

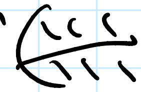
HHH (algebraic links) is even.

Expect for some cables of torus links, (sufficiently large slope...)

(e)  L_k <u>Fact</u>: $L_i L_k = L_k L_i$ in the braid group.

<u>Conj</u>: (a) $HHH(L_1^{a_1} L_2^{a_2} \ldots L_n^{a_n})$ is even provided $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$

(b) Same for $L_1^{a_1} \ldots L_n^{a_n}$ ( ).

(b) Same for $L_1^{a_1} \cdots L_n^{a_n}$ ().

<u>**Note:**</u> If (a) is true, then

$$HY(L_1^{a_1} \cdots L_n^{a_n}) = \bigcap_{i<j} (x_i - x_j,\, y_i - y_j,\, \theta_i - \theta_j)^{a_j}$$

a_j = lk (i-th and j-th component)