Phase Retrieval with Random Illumination

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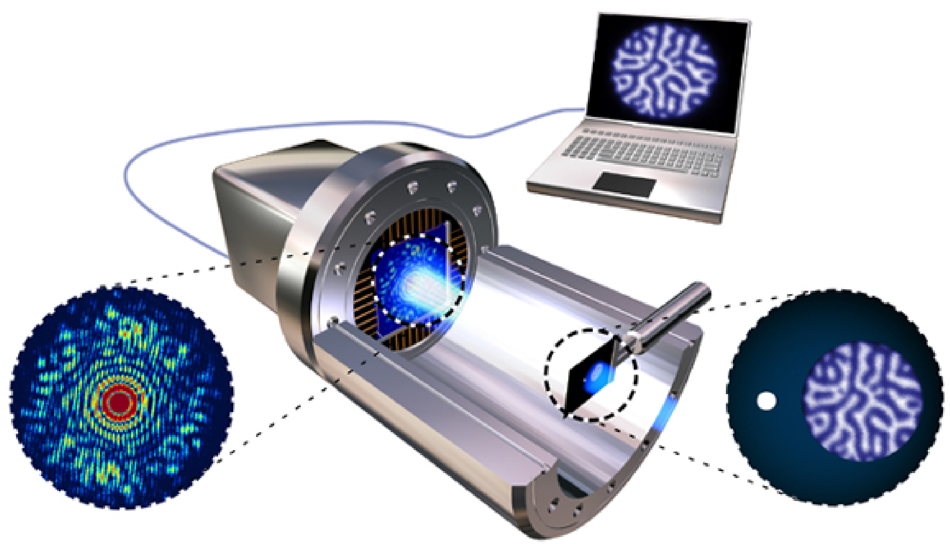
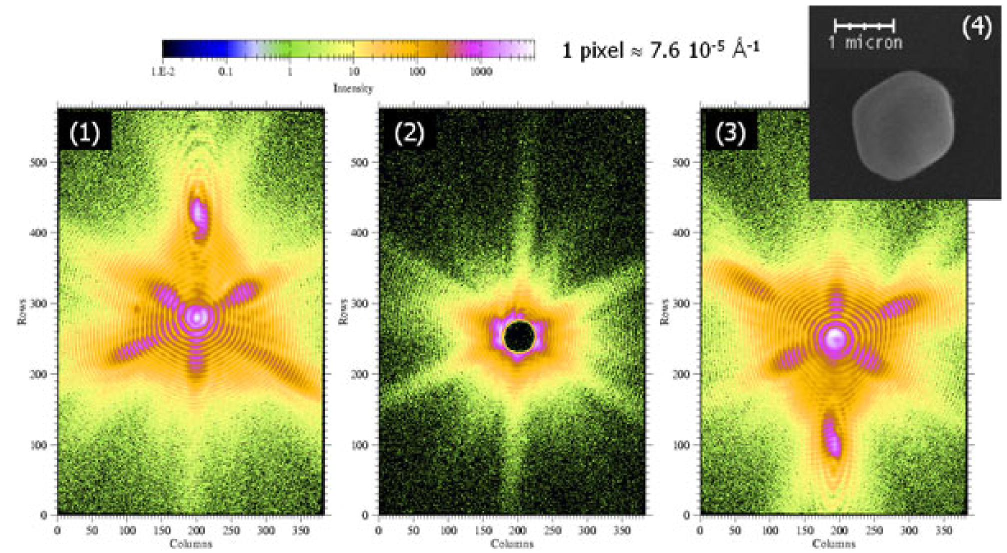
Problem

Reconstruct the object f from the Fourier magnitude $|\Phi f|$.

Why do we care?





X-ray crystallography, single-molecule imaging, astronomy etc.

1985 Nobel Prize in Chemistry for Hauptman and Karle: partial solution of phasing problem.

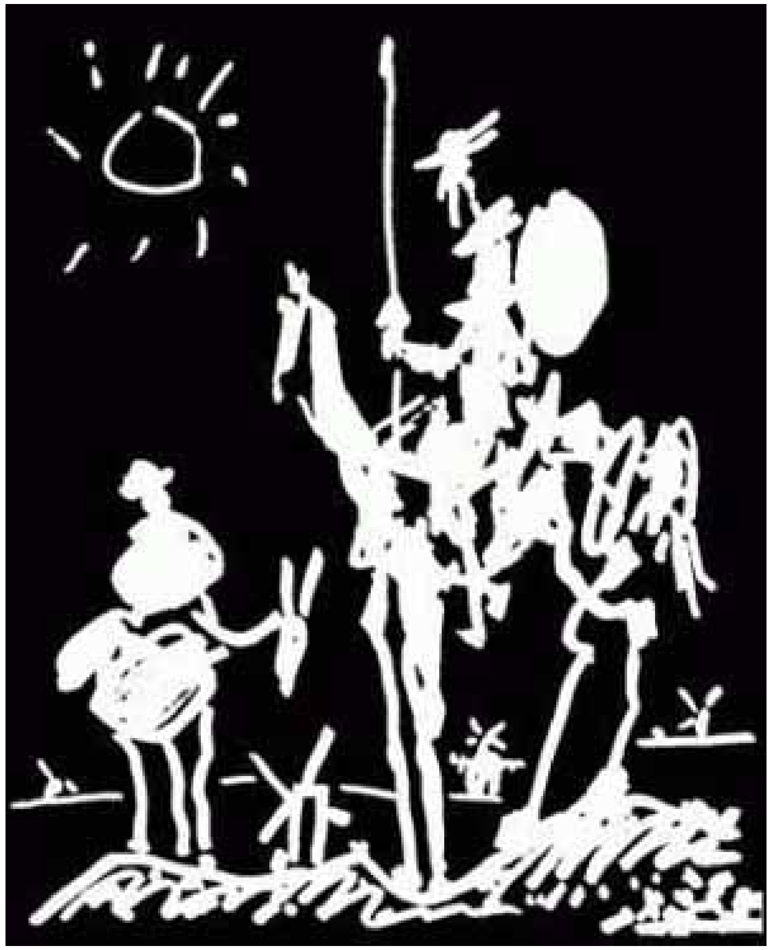
Phasing problem formulation

Discrete finite objects

Let $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$ and $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$.

$$\text{multi-index}: \quad \mathbf{z}^{\mathbf{n}} = z_1^{n_1} z_2^{n_2}$$

Let the *object* be represented by $f(\mathbf{n}), \mathbf{n} \leq \mathbf{N} = (N, N)$.



Binary objects: white $= 1$, black $= 0$.

$$f_L = \text{Lena}$$
$$F_L(\mathbf{w}) = |F_L(\mathbf{w})| e^{\mathrm{i}\theta_L(\mathbf{w})}$$

$$f_B = \text{Barbara}$$
$$F_B(\mathbf{w}) = |F_B(\mathbf{w})| e^{\mathrm{i}\theta_B(\mathbf{w})}$$

$$F_1(\mathbf{w}) = |F_B(\mathbf{w})| e^{\mathrm{i}\theta_L(\mathbf{w})}$$
$$f_1 = |\mathbf{\Phi}^* F_1|$$

$$F_2(\mathbf{w}) = |F_L(\mathbf{w})| e^{\mathrm{i}\theta_B(\mathbf{w})}$$
$$f_2 = |\mathbf{\Phi}^* F_2|$$

$$F_1(\mathbf{w}) = |F_B(\mathbf{w})| e^{\mathrm{i}\theta_L(\mathbf{w})}$$
$$f_1 = |\mathbf{\Phi}^* F_1|$$

$$F_2(\mathbf{w}) = |F_L(\mathbf{w})| e^{\mathrm{i}\theta_B(\mathbf{w})}$$
$$f_2 = |\mathbf{\Phi}^* F_2|$$

Phase retrieval

Discrete finite objects

Let $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$ and $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$.

multi-index : $\mathbf{z}^{\mathbf{n}} = z_1^{n_1} z_2^{n_2}$

Let the object be represented by $f(\mathbf{n}), \mathbf{n} \leq \mathbf{N} = (N, N)$

Fourier transform describes wave propagation

$$F(e^{\mathrm{i}2\pi w_1}, e^{\mathrm{i}2\pi w_2}) = \sum_{\mathbf{n}} f(\mathbf{n}) e^{-\mathrm{i}2\pi \mathbf{n}\cdot\mathbf{w}}$$

Analytic continuation $\Longrightarrow$ z-transform

$$F(\mathbf{z}) = \sum_{\mathbf{n}} f(\mathbf{n}) \mathbf{z}^{-\mathbf{n}}.$$

Discrete phase retrieval problem:

Determine $f(\mathbf{n})$ from Fourier magnitude data

$$|F(\mathbf{w})|, \quad \forall \mathbf{w} = (e^{\mathrm{i}2\pi w_1}, e^{\mathrm{i}2\pi w_2}) \in [0, 1]^2$$

Fourier magnitude data:

$$|F(\mathbf{w})|^2 = \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \sum_{\mathbf{m}} f(\mathbf{m}+\mathbf{n}) f^*(\mathbf{m}) e^{-i2\pi\mathbf{n}\cdot\mathbf{w}}$$
$$= \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \mathcal{C}_f(\mathbf{n}) e^{-i2\pi\mathbf{n}\cdot\mathbf{w}}$$

where

$$\mathcal{C}_f(\mathbf{n}) = \sum_{\mathbf{m}} f(\mathbf{m}+\mathbf{n}) f^*(\mathbf{m})$$

is the autocorrelation function of f.

Fourier magnitude data contain complete information about autocorrelation function.

Sampling Theorem:
supp$(C_f) \subset [-N, N]^2 \Longrightarrow [0,1]^2$ **is reduced to the Nyquist grid**

$$\mathcal{M} = \left\{(k_1, k_2) : k_j = 0, \frac{1}{2N+1}, \frac{2}{2N+1}, \cdots, \frac{2N}{2N+1}\right\}$$

Oversampling ratio

$$\sigma = \frac{\text{Fourier magnitude data number}}{\text{unknown image pixel number}}$$

Standard ratio: $\sigma = 2^d$

Compressed sensing: $\sigma < 2^d$

Trivial ambiguities

Autocorrelation:

$$\mathcal{C}_f(\mathbf{n}) = \sum_{\mathbf{m}+\mathbf{n}\in\mathcal{N}} f(\mathbf{m}+\mathbf{n}) f^*(\mathbf{m})$$

Invariant under:

(i) global phase,

$$f(\mathbf{n}) \longrightarrow e^{\mathrm{i}\theta} f(\mathbf{n}), \quad \text{for some } \theta \in [0, 2\pi],$$

(ii) spatial translation

$$f(\mathbf{n}) \longrightarrow f(\mathbf{n} \oplus \mathbf{m}), \quad \mathbf{n} \oplus \mathbf{m} = \mathbf{n} + \mathbf{m} \ \textbf{mod}(N), \quad \textbf{some } \mathbf{m} \in \mathbb{Z}^2$$

(iii) conjugate inversion (twin image)

$$f(\mathbf{n}) \longrightarrow f^*(\mathbf{N} - \mathbf{n}).$$

Nontrivial ambiguity

THEOREM (Hayes 82, Pitts-Greenleaf 03)

Let the z-transform $F(\mathbf{z})$ of a finite complex-valued sequence $\{f(\mathbf{n})\}$ be given by

$$F(\mathbf{z}) = \alpha \mathbf{z}^{-\mathbf{m}} \prod_{k=1}^{p} F_k(\mathbf{z}), \qquad \mathbf{m} \in \mathbb{N}^2, \alpha \in \mathbb{C}$$

where $F_k, k = 1, ..., p$ are nontrivial irreducible polynomials. Let $G(\mathbf{z})$ be the z-transform of another finite sequence $g(\mathbf{n})$. Suppose $|F(\mathbf{w})| = |G(\mathbf{w})|, \forall \mathbf{w} \in [0,1]^2$. Then $G(\mathbf{z})$ must have the form

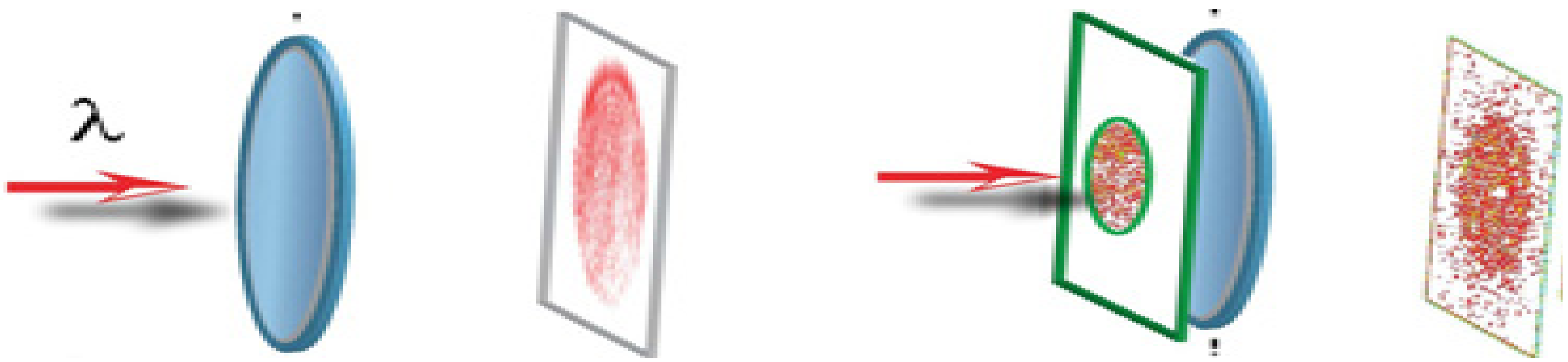
$$G(\mathbf{z}) = |\alpha| e^{\mathrm{i}\theta} \mathbf{z}^{-\mathbf{p}} \left(\prod_{k \in I} F_k(\mathbf{z}) \right) \left(\prod_{k \in I^c} F_k^*(1/\mathbf{z}^*) \right), \qquad \mathbf{p} \in \mathbb{N}^2, \theta \in \mathbb{R}$$

where I is a subset of $\{1, 2, ..., p\}$.

Nontrivial ambiguity: Partial conjugate inversion on factors.

Random illumination

Coded aperture imaging



Diffuser generated speckle pattern: Garcia-Zalevsky-Fixler 05

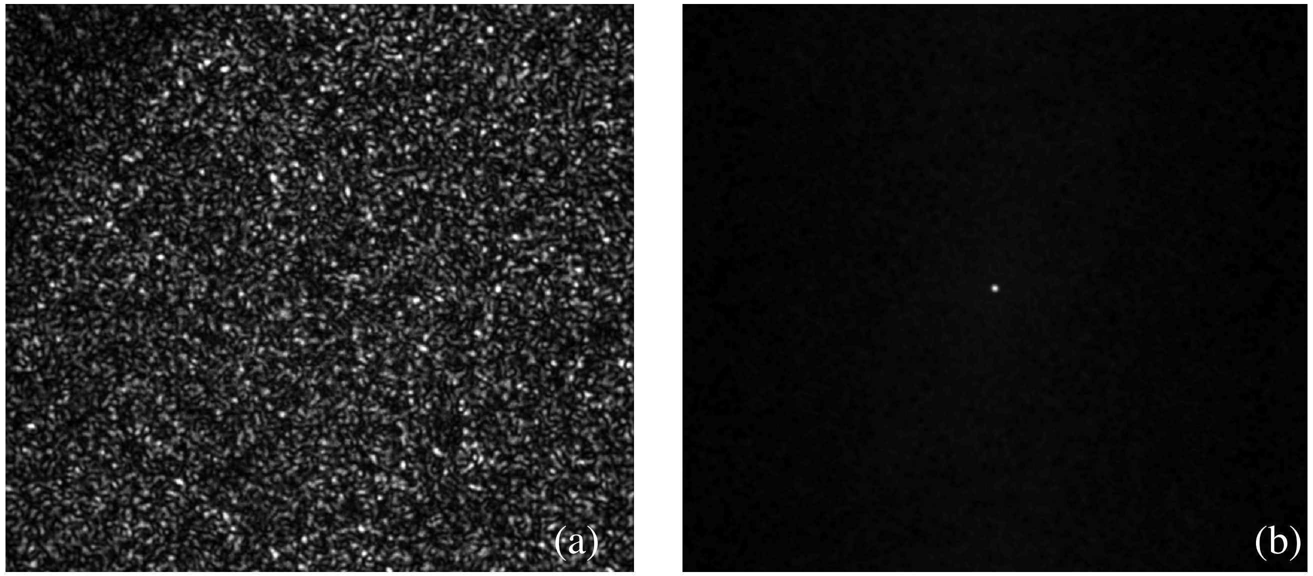
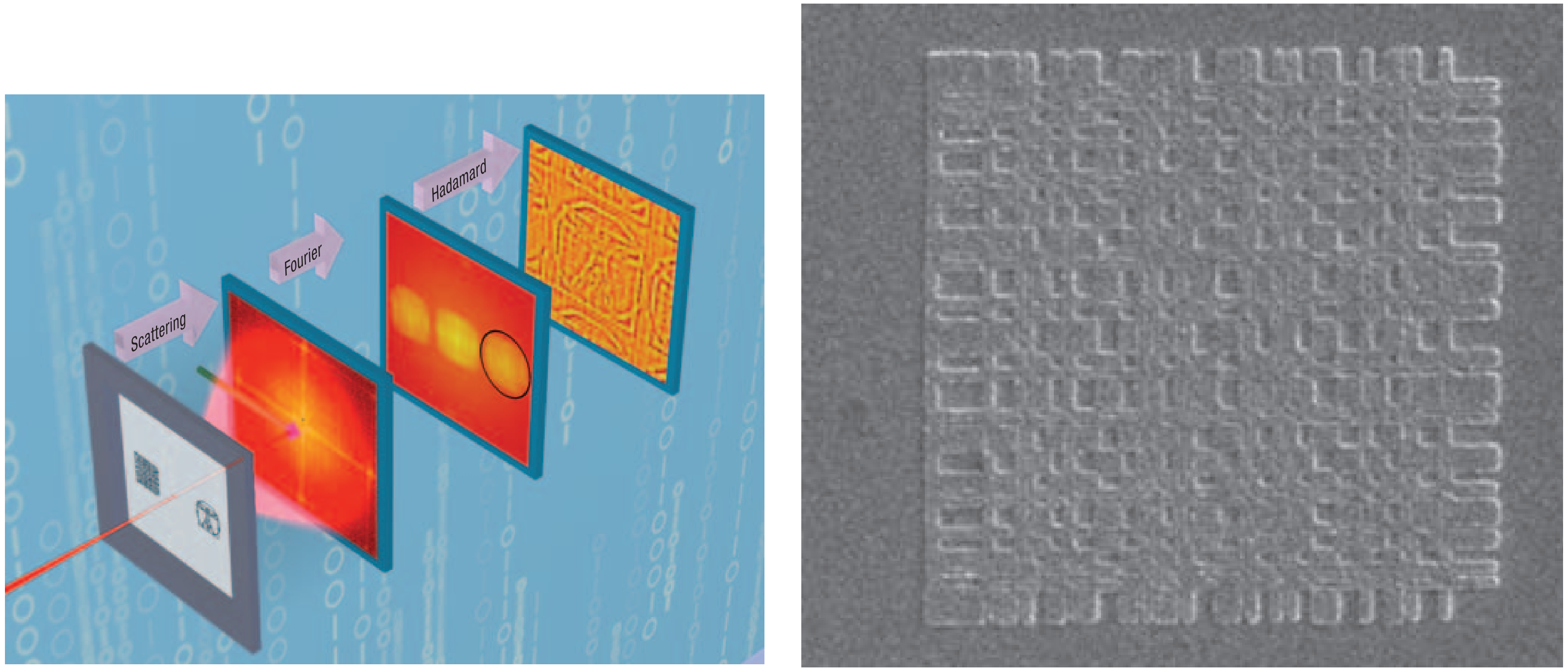


Fig. 2. (a) Encoding speckle pattern. (b) Autocorrelation of the encoding pattern.



X-ray holograhy with URA (Marchesini *et al.* 08).

Coherent X-ray imaging will be a key technique for developing nanoscience and nanotechnology.

One shot imaging: Bright and ultrashort X-ray pulse vaporizes the sample right after the pulse passes the sample.

Random illumination amounts to replacing the original object $f(\mathbf{n})$ by

$$\tilde{f}(\mathbf{n}) = f(\mathbf{n})\lambda(\mathbf{n}) \quad \text{(illuminated object)}$$

Random illumination

$$\tilde{f}(\mathbf{n}) = f(\mathbf{n})\lambda(\mathbf{n}) \quad \text{(illuminated object)}$$

$\lambda(\mathbf{n})$, representing the illumination field, is a known sequence of samples of random variables.

Let $\lambda(\mathbf{n})$ be continuous random variables with respect to the Lebesque measure on $\mathbb{S}^1$ (the unit circle), $\mathbb{R}$ or $\mathbb{C}$.

Case of $\mathbb{S}^1$ can be facilitated by a random phase modulator with

$$\lambda(\mathbf{n}) = e^{\mathrm{i}\phi(\mathbf{n})}$$

where $\phi(\mathbf{n})$ are continuous random variables on $[0, 2\pi]$.
Case of $\mathbb{R}$: random amplitude modulator.
Case of $\mathbb{C}$: both phase and amplitude modulations.

Irreducibility

THEOREM. Suppose that the support of the object $\{f(\mathrm{n})\}$ has rank ≥ 2. Then the the z-transform of the illuminated object $f(\mathrm{n})\lambda(\mathrm{n})$ is irreducible with probability one.

False for rank 1 objects: fundamental thm of algebra

$$\sigma = \frac{\text{Fourier magnitude data number}}{\text{unknown image pixel number}}$$

$$= 2^d$$

Absolute uniqueness

Positivity

THEOREM If $f(\mathrm{n})$ is real and nonnegative for every n then, with probability one, f is determined absolutely uniquely by the Fourier magnitude measurement on the lattice $\mathcal{L}$.

Sector constraint

THEOREM Suppose the phases of the object belong to $[a, b] \subset [0, 2\pi]$. Then the solution to the Fourier phasing problem has a unique solution with probability exponentially close to unity (depending on the sparsity and the phase range $|b - a|$.)

Complex objects w/o constraint

THEOREM. Suppose that $\{\lambda_1(\mathrm{n})\}$ are i.i.d. continuous random variables with respect to the Lebesgue measure on $\mathbb{S}^1$, $\mathbb{R}$ or $\mathbb{C}$ and in addition either one of the following conditions is true.

(i) $\{\lambda_2(\mathrm{n})\}$ are i.i.d. continuous random variables with respect to the Lebesgue measure on $\mathbb{S}^1$, $\mathbb{R}$ or $\mathbb{C}$ and $\{\lambda_2(\mathrm{n})\}$ are independent of $\{\lambda_1(\mathrm{n})\}$.

(ii) $\{\lambda_2(\mathrm{n})\}$ are deterministic.

Then with probability one $f(\mathrm{n})$ is uniquely determined, up to a constant phase factor, by the Fourier magnitude measurements with two illuminations λ_1 and λ_2.

Alternated projections

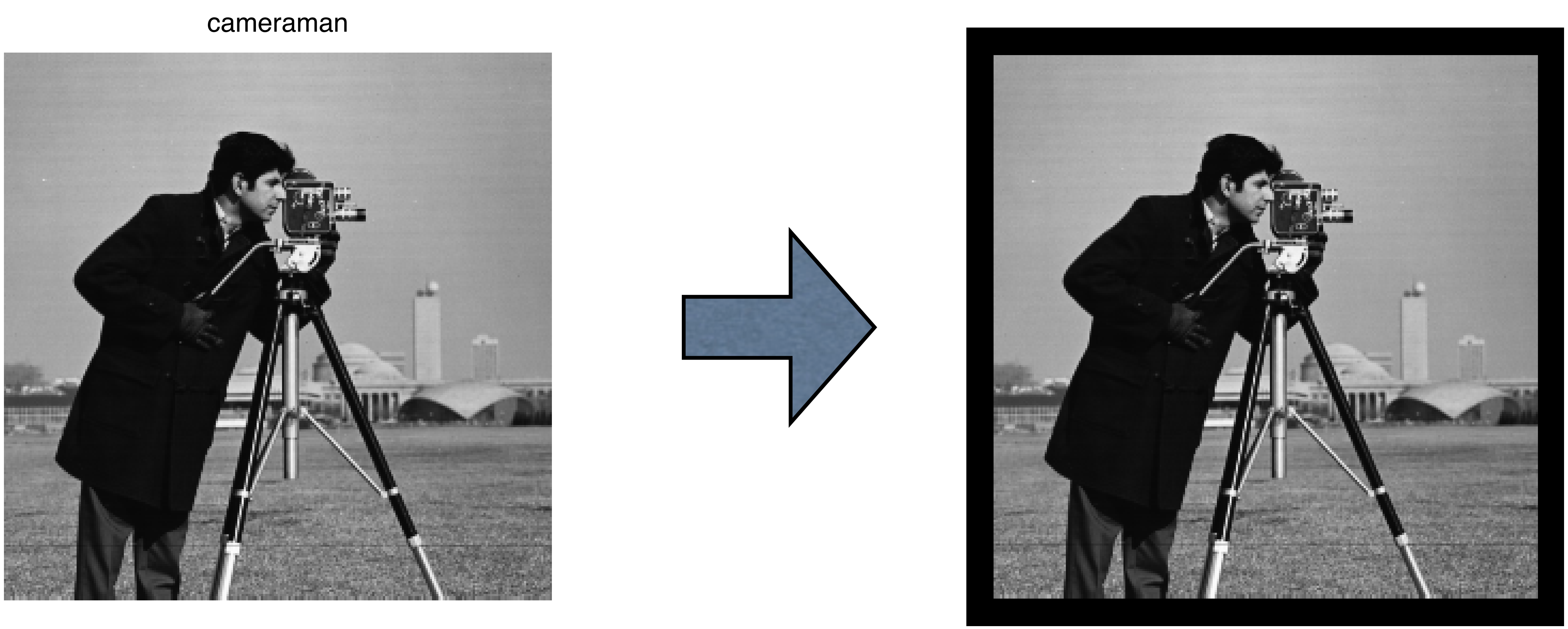
Gerchberg-Saxton; Error Reduction (Fienup)

$$
\begin{array}{ccccc}
f_{k+1}/f_k & \xrightarrow{\Lambda} & g_k & \xrightarrow{\Phi} & G_k \\
\uparrow \mathcal{P}_{\mathrm{o}} & & & & \downarrow \mathcal{T} \\
f'_k & \xleftarrow[\Lambda^{-1}]{} & g'_k & \xleftarrow[\Phi^{-1}]{} & G'_k
\end{array}
$$

Oversampling method (Miao et al.)

$$\sigma = \frac{\text{image pixel number} + \text{zero-padding pixel number}}{\text{image pixel number}}.$$

Zero-padding



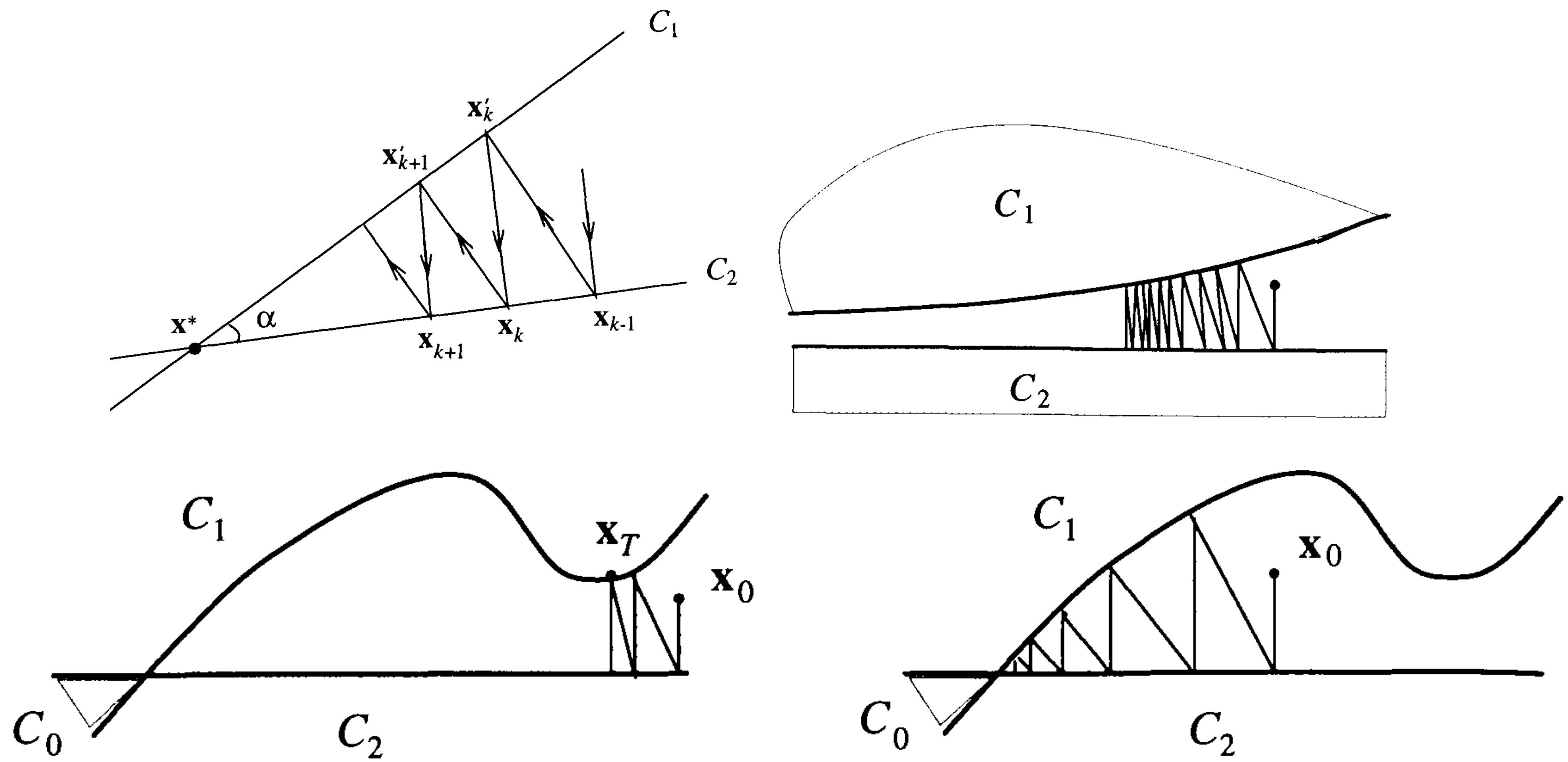
Multiple illuminations

$$\mathcal{P}_1 = \Lambda_1^{-1}\Phi^{-1}\mathcal{T}_1\Phi\Lambda_1$$

$$\mathcal{P}_2 = \Lambda_2^{-1}\Phi^{-1}\mathcal{T}_2\Phi\Lambda_2.$$

$$f_{k+1} = \mathcal{P}_\mathrm{o}\mathcal{P}_2\mathcal{P}_1 f_k.$$

Error Reduction (Gerchberg-Saxton)



Bregman 65: convex constraints $\Longrightarrow$ convergence to a feasible solution.

Fourier magnitude data are a non-convex constraint!

Nonconvexity or nonuniqueness ?

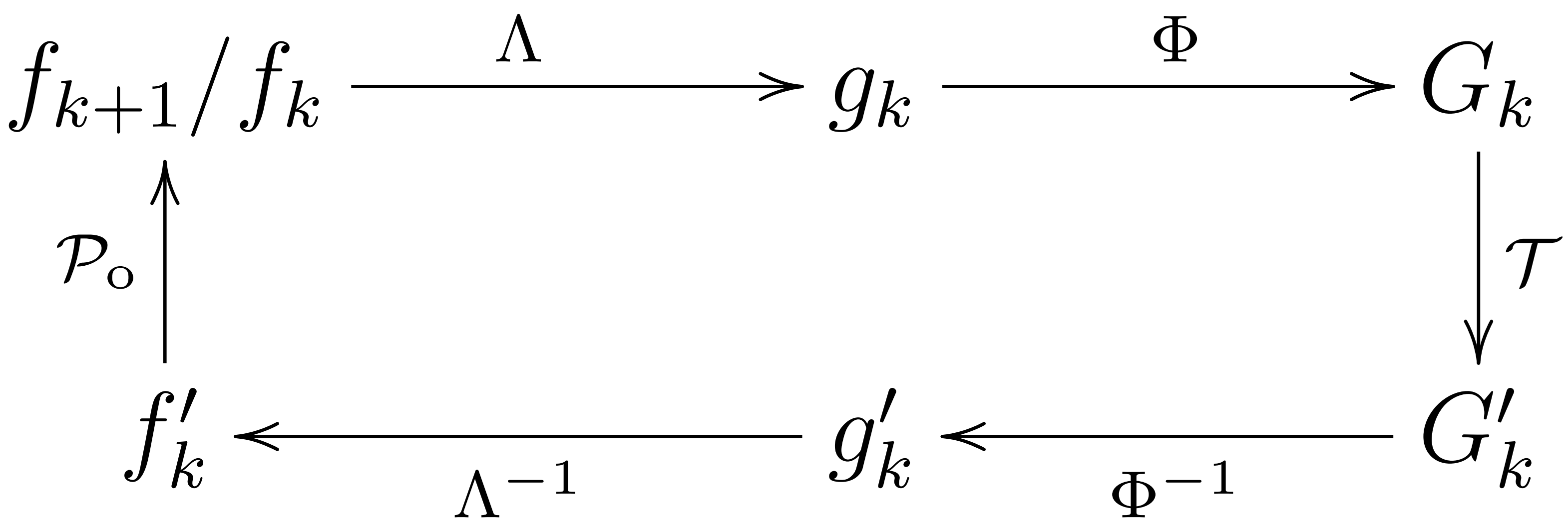
Convergence

Theorem 4. *Let* $f \in \mathcal{C}(\mathcal{N})$ *be an array with* $f(\mathbf{0}) \neq 0$ *and rank* ≥ 2. *Let* $\lambda(\mathbf{n})$ *be i.i.d. continuous random variables on* $\mathbb{S}^1$. *Let the Fourier magnitude be sampled on* $\mathcal{L}$. *Let* h *be a fixed point of* $\mathcal{P}_{\mathrm{o}}\mathcal{P}_{\mathrm{f}}^{\theta}$ *such that* $\mathcal{P}_{\mathrm{f}}^{\theta}h$ *satisfies the zero-padding condition.*

(a) *If* f *is real-valued,* $h = \pm f$ *with probability one,*

(b) *If* f *satisfies the sector condition of Theorem 2, then* $h = e^{i\nu} f$, *for some* ν, *and satisfies the same sector constraint with probability at least* $1 - |\mathcal{N}|(\beta - \alpha)^{[\![S/2]\!]}(2\pi)^{-[\![S/2]\!]}$.

Hybrid-Input-Output (HIO)



$$\begin{aligned} \Re(f_{k+1}(\mathbf{n})) &= \Re(f'_k(\mathbf{n})) \\ \Im(f_{k+1}(\mathbf{n})) &= \Im(f_k(\mathbf{n})) - \beta \cdot \Im(f'_k(\mathbf{n})), \end{aligned}$$

Real-valued objects

Error metrics

Relative error

$$e(\hat{f}) = \begin{cases} \|f - \hat{f}\| / \|f\| \\ \min\limits_{\nu \in [0, 2\pi)} \|f - e^{i\nu}\hat{f}\| / \|f\| \end{cases}$$

Relative residual

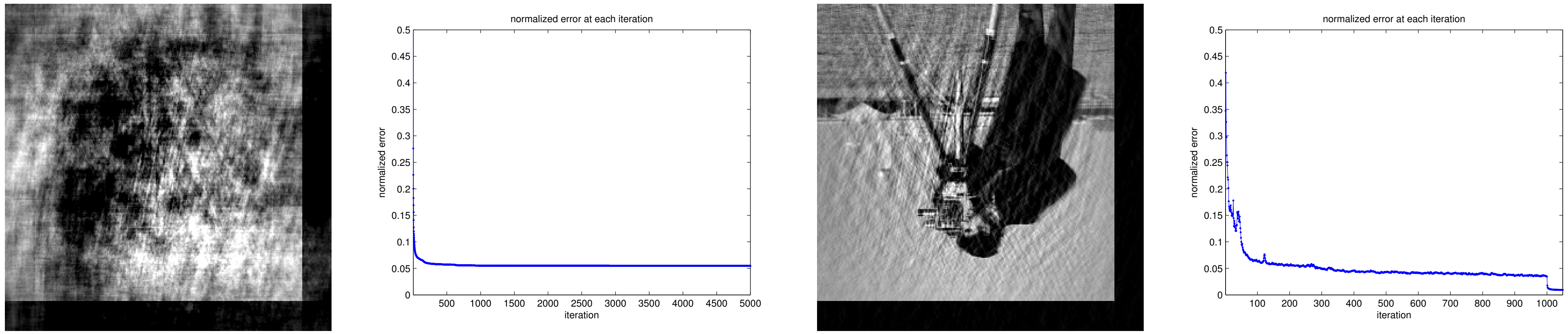
$$r(\hat{f}) = \frac{\| \, Y - |\Phi \Lambda \mathcal{P}_{\mathrm{o}}\{\hat{f}\}| \, \|}{\|Y\|}$$

(a)

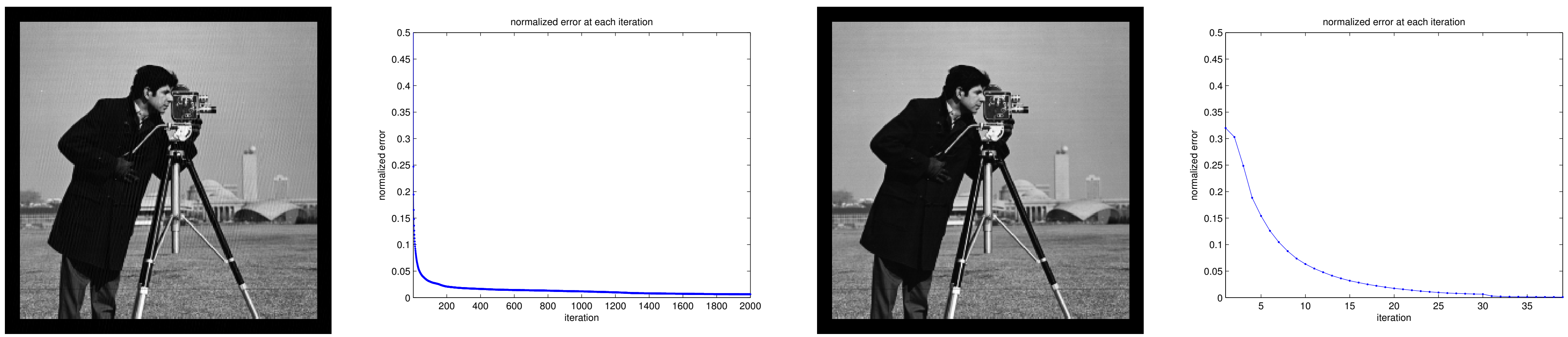
(b)

269 x 269

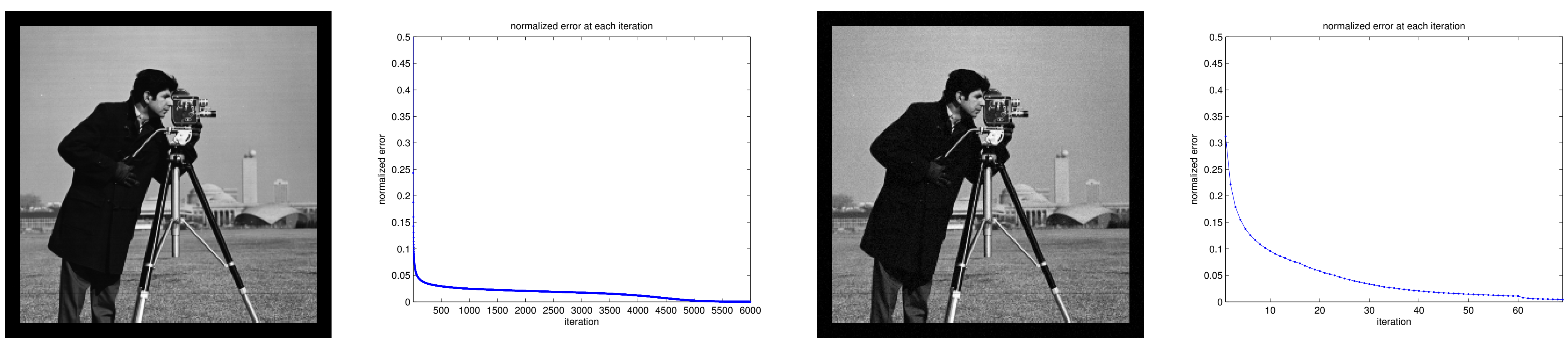
200 x 200



(a) (b) (c) (d)

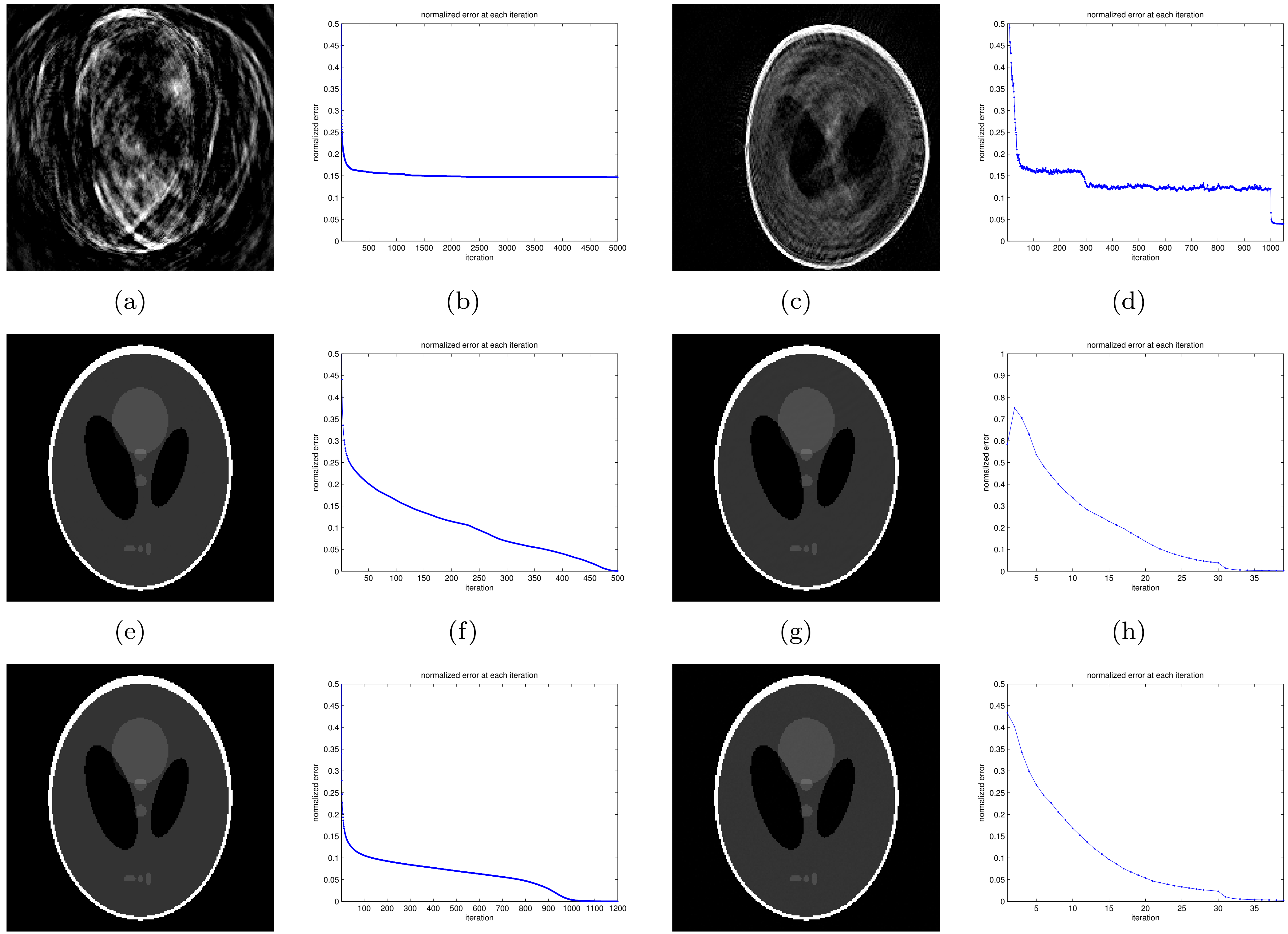


(e) (f) (g) (h)



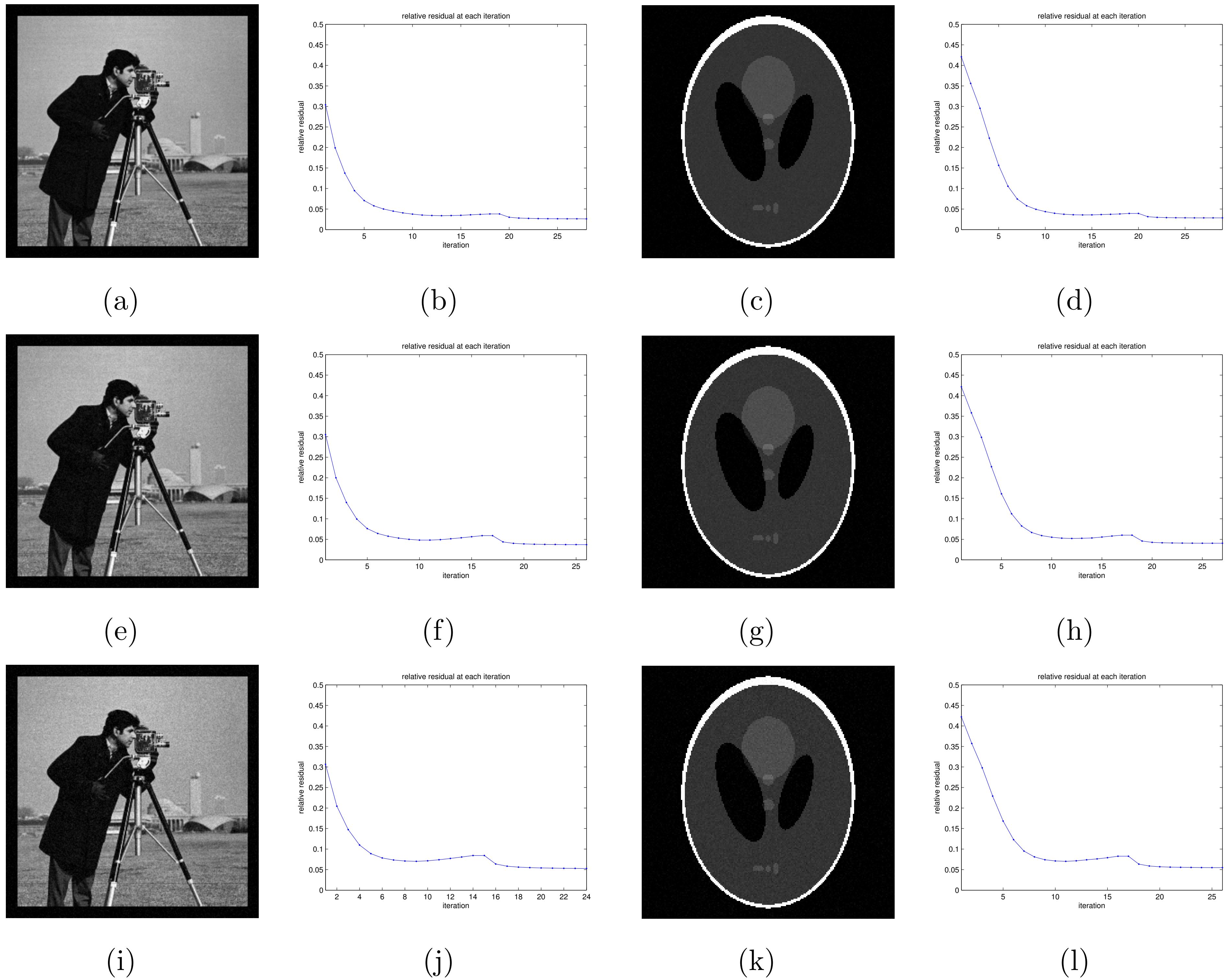
(i) (j) (k) (l)

(e)-(h) Low resolution 40 x 40 block illumination with OR=2
(i)-(l) High resolution illumination with OR=1

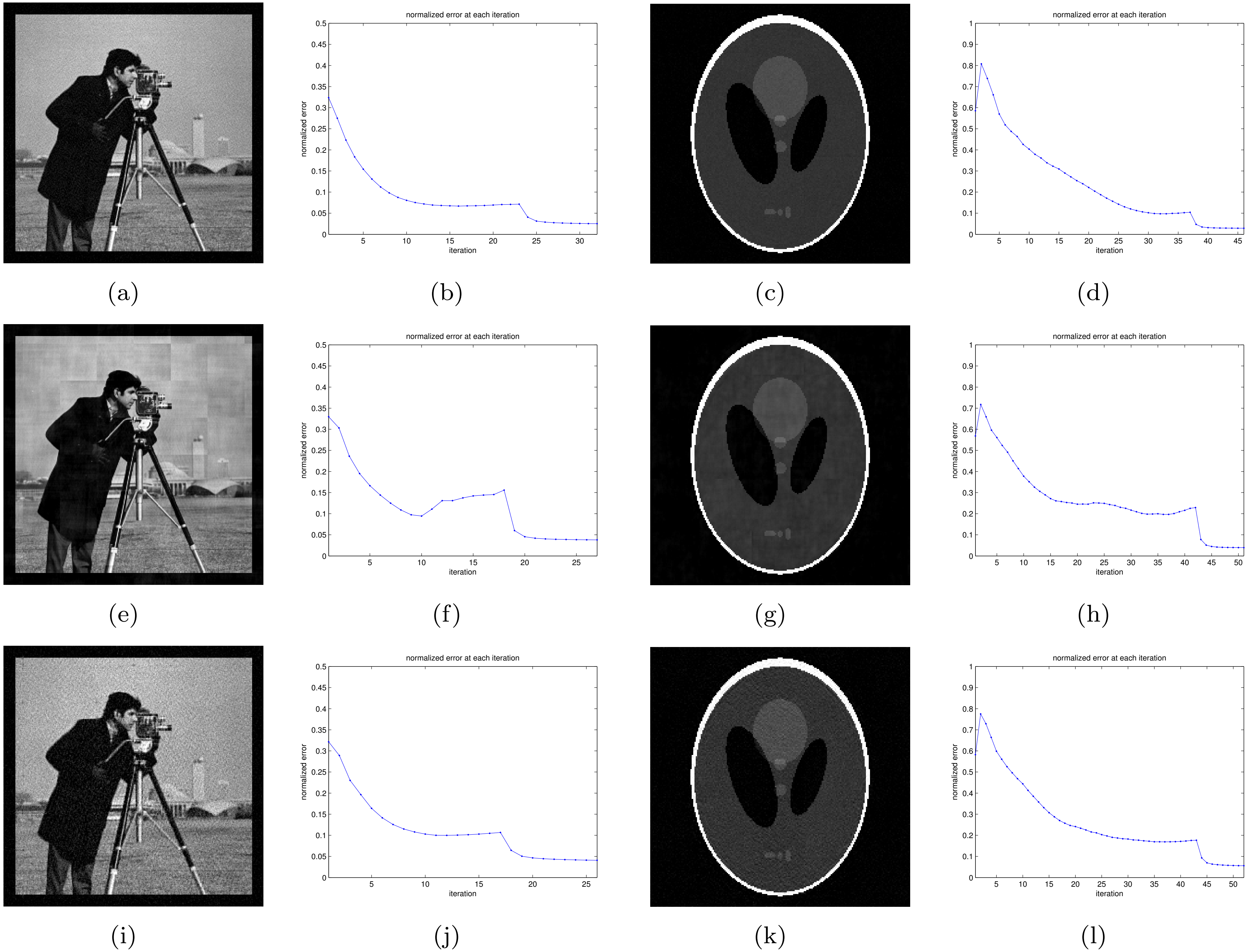


(a) (b) (c) (d)
(e) (f) (g) (h)
(i) (j) (k) (l)

(e) - (h) Low resolution 40 x 40 block illumination with OR=2
(i) - (l) High resolution illumination with OR=1

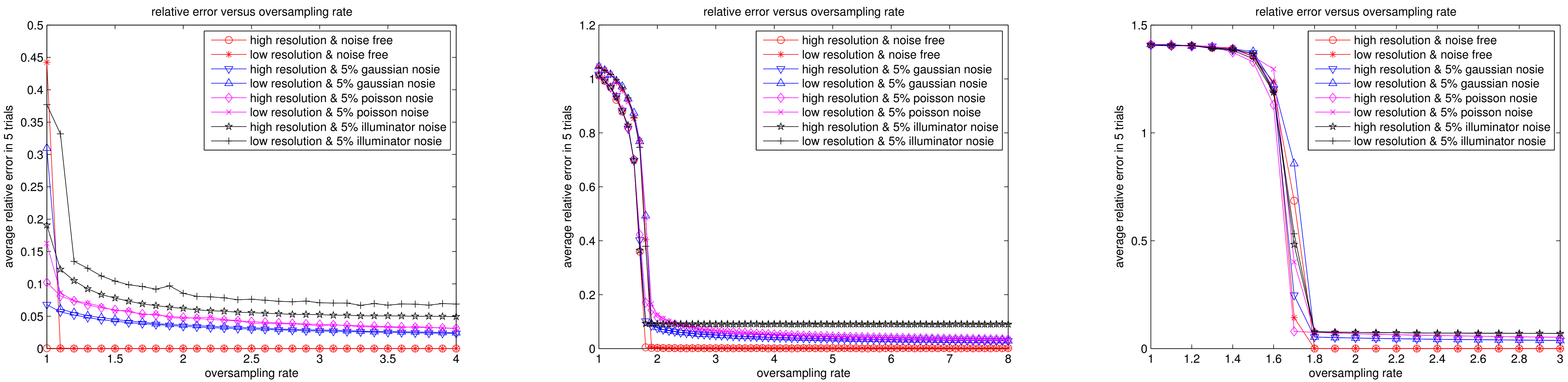


High resolution illumination with 5% Gaussian, Poisson and illuminator errors



Low resolution illumination with 5% Gaussian, Poisson and illuminator errors

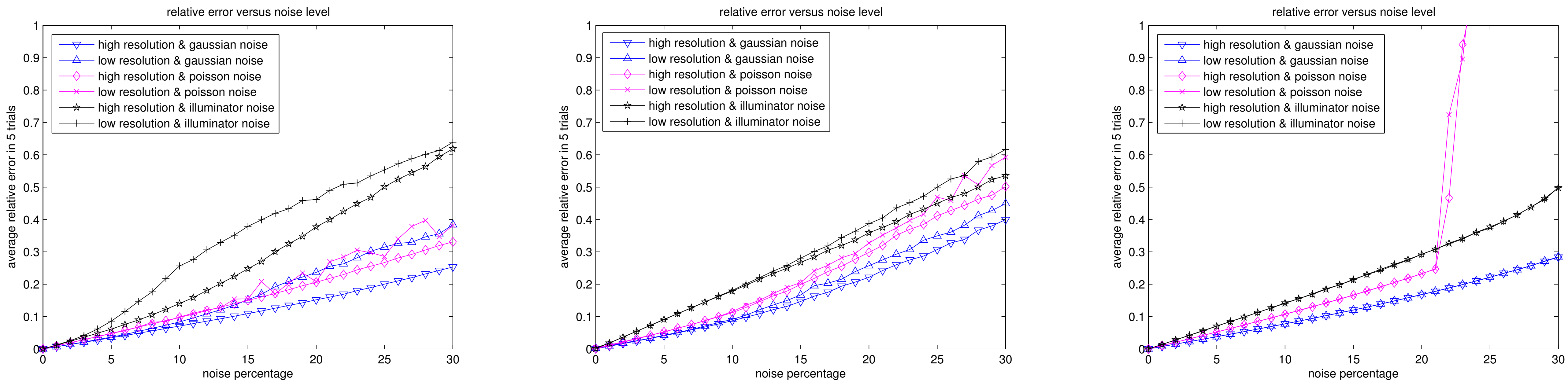
Compressed measurement



(a) (b) (c)

(a) real-valued
(b) positive real & imaginary parts
(c) no constraint

Noise stability



(a) (b) (c)

(a) real-valued objects
(b) positive real & imaginary parts
(c) no constraint

Conclusions

- Random illumination as enabling tool for phase retrieval.
- Absolute uniqueness
- Fast convergence
- OR =1 (real) or 2 (complex)
- Proof of convergence: HIO?
- References:

- A. Fannjiang **Absolute uniqueness in phase retrieval with random illumination** Inverse Problems 2012 (arXiv:1110.5097)
- A. Fannjiang and W. Liao **Phase retrieval with random phase illumination** arXiv:1206.1001