ANOMALOUS DIFFUSION IN RANDOM FLOWS

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1. Introduction. The simplest model of turbulent transport is the random motion of Brownian particles passively convected by random, incompressible velocity fields. The particle path is the solution of the stochastic differential equation

$$d\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t))dt + \sqrt{2\kappa}d\boldsymbol{w}(t) \tag{1.1}$$

where $\boldsymbol{w}(t)$ is the standard Brownian motion, $\kappa > 0$ is the molecular diffusion and the velocity field $\boldsymbol{u}$ is random stationary, divergence free:

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x}) = 0, \tag{1.2}$$

and has zero mean

$$\langle \boldsymbol{u} \rangle = 0. \tag{1.3}$$

Here and below $\langle \cdot \rangle$ stands for the ensemble average. The concentration $\rho(\boldsymbol{x}, t)$ of passive scalar particles, whose sample path $\boldsymbol{x}(t)$ defined by (1.1), satisfies the convection-diffusion equation

$$\frac{\partial \rho(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x}) \cdot \nabla \rho(\boldsymbol{x}, t) = \kappa \Delta \rho(\boldsymbol{x}, t). \tag{1.4}$$

The coupling between the molecular diffusion and the randomness velocity gives rise to many interesting, long time and large scale behaviors of solutions of (1.1) and (1.4). The object of interest is the long time, large space scaling laws which take many different forms. The simplest one is perhaps

$$\sqrt{\langle \boldsymbol{E}(|\boldsymbol{x}|^2(t)) \rangle} \sim t^p, \quad \text{as } t \to \infty \tag{1.5}$$

with the root-mean-square displacement expressed as a function of time, where $\boldsymbol{E}$ denotes the average w.r.t. the Brownian motion. The problem is to compute the exponent p from (1.1) or (1.15) which is often difficult to do analytically (cf. [14],[4]).

In general scaling laws can be viewed roughly as the relationship between the space scale and the time scale, whichever is used as the parameter, associated with the solutions of equation (1.1) or (1.4). In this spirit, (1.5) can be interpreted as

$$\text{space} \sim (\text{time})^p. \tag{1.6}$$

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When $p = 1/2$, it is called *normal diffusion*; $p > 1/2$, *superdiffusion*; $p < 1/2$, *subdiffusion*. Super- and sub-diffusions are also called *anomalous* diffusions. It will be clear later that subdiffusion does not occur in incompressible flows – the diffusivity is always enhanced – namely,

$$p \geq 1/2. \tag{1.7}$$

In the present paper, we take a different approach: We shall study the asymptotics of the *box diffusivity* matrix σ_n of scale n as n tends to infinity. The box diffusivity $\sigma_n(\boldsymbol{e}_1)$ of scale n in the direction of $\boldsymbol{e}_1$ may be defined as the energy integral

$$\sigma_n(\boldsymbol{e}_1) = \sigma_n \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = \frac{\kappa}{n^d} \int_{[0,n]^d} \nabla\rho_1 \cdot \nabla\rho_1 \, d\boldsymbol{x} \tag{1.8}$$

for the steady-state problem

$$\kappa\Delta\rho_1(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x}) \cdot \nabla\rho_1(\boldsymbol{x}) = 0, \quad \text{in } [0,n]^d \tag{1.9}$$

with the mean of the *periodic* concentration gradient maintained at $\boldsymbol{e}_1$

$$\frac{1}{n^d} \int_{[0,n]^d} \nabla\rho_1 = \boldsymbol{e}_1. \tag{1.10}$$

Because of the periodicity of $\nabla\rho_1$ and (1.10) we can write $\rho_1(\boldsymbol{x}) = x_1 + \rho(\boldsymbol{x})$ with a periodic function ρ. The local linear constitutive law

$$\boldsymbol{D}_1 = \kappa\nabla\rho_1 + \boldsymbol{u}\rho_1 \tag{1.11}$$

relates the concentration ρ_1 to the flux $\mathbf{D}_1$. The average flux relates again to the average concentration gradient, which is $\boldsymbol{e}_1$, linearly

$$\frac{1}{n^d} \int_{[0,n]^d} \boldsymbol{D}_1 = \sigma_n \boldsymbol{e}_1, \tag{1.12}$$

where the proportionality is exactly the box diffusivity σ_n, can be seen by the energy equality for (1.9).

Our strategy is to study the asymptotic scaling law of the energy integral (1.8)

$$\langle\sigma_n(\boldsymbol{e}_1)\rangle = \frac{\kappa}{n^d} \int_{[0,n]^d} \langle\nabla\rho_1 \cdot \nabla\rho_1\rangle \, d\boldsymbol{x} \sim n^q, \quad \text{as } n \to \infty \tag{1.13}$$

using a variational method and its dual. If the scaling exponent q is consistent with the scaling exponent p then, in view of (1.6), we expect that

$$q = 2 - 1/p \tag{1.14}$$

following simple dimensional analysis.

Another quantity, the mean exit time $\rho(\boldsymbol{x}) = \boldsymbol{E}_{\boldsymbol{x}}(\tau_n)$, where τ_n is the exit time from the box starting at $\boldsymbol{x}$ at $t = 0$, can be studied the same way. The function $\rho(\boldsymbol{x})$ satisfies the PDE

$$\kappa\Delta\rho(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x}) \cdot \nabla\rho(\boldsymbol{x}) = -1, \quad \text{in } [0, n]^d \tag{1.15}$$

with zero Dirichlet data on the boundary of the box $[0, n]^d$. Suppose that the average value satisfies the scaling law

$$\frac{1}{n^d} \int_{[0,n]^d} \langle\rho(\boldsymbol{x})\rangle \, d\boldsymbol{x} \sim n^r, \quad \text{as } n \to \infty \tag{1.16}$$

then by the energy estimate for (1.15) we have the same scaling law for the energy integral

$$\frac{\kappa}{n^d} \int_{[0,n]^d} \langle\nabla\rho \cdot \nabla\rho\rangle \, d\boldsymbol{x} \sim n^r, \quad \text{as } n \to \infty. \tag{1.17}$$

By dimensional analysis, we expect from (1.6) that

$$r = 1/p. \tag{1.18}$$

Both above problems can be put into the form

$$\kappa\Delta\rho(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x}) \cdot \nabla\rho(\boldsymbol{x}) = f(\boldsymbol{x}) \tag{1.19}$$

where $f(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{e}_1$ for the box diffusivity and $f(\boldsymbol{x}) = -1$ for the mean exit time. The boundary condition is the periodic condition for the box diffusivity and the homogeneous Dirichlet condition for the mean exit time. In general the forcing term $f(\boldsymbol{x})$ can be any stationary random functions which correspond to various ways of probing the system. Suppose that their corresponding energy integrals have scaling laws, such as (1.13) and (1.17), then we have the whole collection of scaling exponents associated with the steady state problem (1.19) with forcing. Except for the case of normal diffusion $p = 1/2$ (cf. [8]), we do not know if all the scaling exponents are consistent and/or can be reduced to the exponent p in (1.5) in the sense of dimensional analysis as carried out for the box diffusivity and the mean exit time.

In this paper, we focus on the scaling exponent of the box diffusivity because the corresponding forcing term can be absorbed by imposing the mean gradient condition (1.10) which is easiest to deal with by the variational duality argument. We will continue to write the exponent q in the form

$$q = 2 - 1/\lambda \tag{1.20}$$

for some $\lambda \geq 1/2$, and compare λ to the exponent p in (1.5) since they have the same unit and should be the same on physical grounds.

To facilitate the variational approach, we write the equation (1.9) in the divergence form

$$\nabla \cdot (\kappa I + \Psi) \nabla \rho_1 = 0 \tag{1.21}$$

using the stream matrix $\Psi(\boldsymbol{x})$ defined by

$$\nabla \cdot \Psi = \boldsymbol{u}. \tag{1.22}$$

In three dimension, the stream matrix $\Psi(\boldsymbol{x})$ is related to the vector potential $\boldsymbol{\psi}(\boldsymbol{x}) = (\psi_1(\boldsymbol{x}), \psi_2(\boldsymbol{x}), \psi_3(\boldsymbol{x}))$, $\nabla \times \boldsymbol{\psi}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})$, in the following way

$$\Psi(\boldsymbol{x}) = \begin{pmatrix} 0 & -\psi_3 & \psi_2 \\ \psi_3 & 0 & -\psi_1 \\ -\psi_2 & \psi_1 & 0 \end{pmatrix}. \tag{1.23}$$

In two-dimension, the stream matrix takes the form

$$\Psi(\boldsymbol{x}) = \begin{pmatrix} 0 & -\psi(\boldsymbol{x}) \\ \psi(\boldsymbol{x}) & 0 \end{pmatrix}, \tag{1.24}$$

where $\psi(\boldsymbol{x})$ is the usual stream function, $\nabla^{\perp}\psi = \boldsymbol{u}$, in fluid mechanics. To determine Ψ uniquely we demand that the vector potential $\boldsymbol{\psi}$ be divergence-free

$$\nabla \cdot \boldsymbol{\psi} = 0 \tag{1.25}$$

and $\boldsymbol{\psi}(0) = 0$. In this paper, we consider the stream matrices that satisfy the full discrete symmetry (such as $\pi/2$-rotational symmetry in two dimension , see Section 4).

Notice that the stream matrix is in general *not* stationary unless the velocity field has fast decaying correlation and the dimension is three or higher (cf. Section 2 for the precise condition). In fact, our analysis shows that scaling exponent q is directly related to the far-field behavior of non-stationary stream matrix Ψ and is given by

$$q = \mu \tag{1.26}$$

where μ is the *growth index* of the stream matrix

$$\sqrt{\langle |\Psi(n\boldsymbol{x}) - \Psi(0)|^2 \rangle} \sim n^{\mu}, \quad \mu \leq 1 \tag{1.27}$$

as $n \to \infty, \forall \boldsymbol{x} \neq 0$. We note that dimensionally

$$[\Psi] = \frac{[\text{space}]^2}{[\text{time}]} \tag{1.28}$$

which is the same as that of the box diffusivity and hence (1.26) is dimensionally correct. The results (1.26) is independent of the dimension $d \geq 3$.

For $d = 2$, certain restriction (cf. the consistency condition (4.15)) on the index μ applies (see Section 4). The consistency condition (4.15) is also required for the exponent q to be well defined in two dimensions.

As a result of (1.26) and (1.20), we have

$$\lambda = \frac{1}{2-\mu}. \tag{1.29}$$

When the stream matrix has the logarithmic anomaly

$$\langle |\Psi(n\boldsymbol{x})|^2 \rangle \sim \log n, \tag{1.30}$$

(which is typical in two dimensions), we have

$$\langle \sigma_n \rangle \sim \sqrt{\log n}. \tag{1.31}$$

For *isotropic* random flows (hence possessing the full discrete symmetry) with velocity spectrum

$$\hat{R}_{ij}(\boldsymbol{k}) \sim \frac{1}{|\boldsymbol{k}|^{2\nu+d-2}}(\delta_{i,j} - \frac{k_i k_j}{|\boldsymbol{k}|^2}), \quad |\boldsymbol{k}| \ll 1, \quad \nu \leq 1 \tag{1.32}$$

the two-point correlation $R_{ij}(\boldsymbol{x})$ is asymptotically

$$R_{ij}(\boldsymbol{x}) \sim \frac{1}{(1+|\boldsymbol{x}|)^{2(1-\nu)}}, \quad \nu \leq 1, \tag{1.33}$$

for $|\boldsymbol{x}| \gg 1$, and the growth index is given by the formula

$$\mu = \begin{cases} \nu, & \text{for } 0 < \nu \leq 1, \ d \geq 2 \\ 0, & \text{for } \nu < 0, \ d \geq 3 \end{cases} \tag{1.34}$$

which, for $d = 2$, is subject to the consistency condition (4.15). For $\nu > 1$ in (1.32), (1.33), the velocity field is not L^2-stationary without a far field cut-off because the correlation diverges at large distances. Introducing a cut-off at scale $L \gg 1$, we write

$$\hat{R}_{ij}(\boldsymbol{k}) \sim \begin{cases} \frac{1}{|\boldsymbol{k}|^{2\nu+d-2}}(\delta_{i,j} - \frac{k_i k_j}{|\boldsymbol{k}|^2}), & |\boldsymbol{k}| \geq 1/L \\ 0, & |\boldsymbol{k}| < 1/L. \end{cases} \tag{1.35}$$

In this case, the growth index μ for $n \leq L$ in (1.27) is one

$$\mu = 1, \quad \nu > 1 \tag{1.36}$$

and the asymptotics (1.27) carries a large coefficient of order $L^{\nu-1}$, namely

$$\sqrt{\langle |\Psi(n\boldsymbol{x}) - \Psi(0)|^2 \rangle} \sim nL^{\nu-1} \tag{1.37}$$

following a calculation similar to (3.6)-(3.9) (see Section 3).

Thus from (1.29), (1.34) and (1.36) we have

$$\lambda = \begin{cases} 1, & \text{for } \nu > 1, \quad d \geq 2 \\ \frac{1}{2-\nu}, & \text{for } 0 < \nu \leq 1, \quad d \geq 2 \\ 1/2, & \text{for } \nu < 0, \quad d \geq 3 \end{cases} \tag{1.38}$$

which, for $d = 2$, is subject to the consistency condition (4.15). The large coefficient in (1.37) results in a similar large coefficient for σ_n

$$\langle \sigma_n \rangle \sim n L^{\nu - 1}, \quad \nu > 1. \tag{1.39}$$

For isotropic flows in the borderline case of either

$$\nu = 0, \quad d \geq 3, \tag{1.40}$$

or

$$\nu \leq 0, \quad d = 2, \tag{1.41}$$

the logarithmic anomaly (1.31) holds.

Our results (1.38) and (1.31) are consistent with existing results on scaling exponents in *incompressible* flows in the regime $\nu \leq 1$ by totally different approaches (e.g. [3], [4], [9], [14],[16]). In particular, (1.31) is consistent with the result (cf. [9], [14])

$$\langle E(\boldsymbol{x}^2(t)) \rangle \sim t\sqrt{\log t} \tag{1.42}$$

in the case of (1.40) or (1.41). In the regime $\nu > 1$, however, our analysis suggests that no super-ballistic scaling, i.e., $q > 1$, can be produced in the class of *steady, square integrable, stationary velocity fields* given by the truncated spectrum (1.35). In particular, Richardson's law, corresponding to $q = 4/3$, can not be explained by the model given by the spectrum (1.35) for any ν, contrary to what was proposed in some literature (e.g. [4]). For more discussion on this point, see Section 5.

The full discrete symmetry is more general than isotropy. As examples of anisotropic flows with the discrete symmetry, we consider in Section 5 three dimensional flow formed by superposition of random channel flows; the two dimensional analog, the well known Manhattan model has a growth index $\mu = 1/2$ and the corresponding scaling exponent

$$\lambda = 2/3. \tag{1.43}$$

Because the size of κ does not affect our analysis, so long as $\kappa > 0$, we set $\kappa = 1$ for simplicity.

The present report is a summary of the more detailed paper [6] to appear elsewhere.

2. Stationary, square integrable stream matrix implies normal diffusion. Sometimes it is more convenient to work with the rescaled objects

$$\rho_n(\boldsymbol{x}) = n^{-1}\rho(n\boldsymbol{x}), \quad \Psi_n(\boldsymbol{x}) = \Psi_n(n\boldsymbol{x}) \tag{2.1}$$

so as to put the equation for the box diffusivity $\sigma_n(\boldsymbol{e})$ in the direction $\boldsymbol{e}$ in the form

$$\nabla \cdot (I + \Psi_n(\boldsymbol{x}))\, \nabla \rho_n(\boldsymbol{x}) = -\nabla \cdot \Psi_n(\boldsymbol{x}) \cdot \boldsymbol{e}, \quad \text{in } [0,1]^d. \tag{2.2}$$

We note that the scaling (2.1) preserves the energy integral

$$\int_{[0,1]^d} \nabla \rho_n \cdot \nabla \rho_n = \frac{1}{n^d} \int_{[0,n]^d} \nabla \rho \cdot \nabla \rho. \tag{2.3}$$

An energy estimate for (2.2) yields a simple upper bound on the integral

$$\int_{[0,1]^d} d\boldsymbol{x}\ \langle \nabla \rho_n \cdot \nabla \rho_n \rangle \le c \langle \|\Psi_n\|^2_{L^2([0,1]^d)} \rangle = c \| \langle |\Psi_n|^2 \rangle \|^2_{L^2([0,1]^d)}. \tag{2.4}$$

The upper bound in (2.4) implies that normal diffusion occurs when the stream matrix is stationary, square integrable

$$\langle \Psi_{ij}^2 \rangle \le C < \infty, \quad \forall i,j \tag{2.5}$$

where the constant C is independent of scale n. In this case,the right side of (2.4) is independent of scale n.

Below we give a sufficient condition on the velocity field for which a stationary, square integrable stream matrix can be constructed.

First, if Ψ is stationary, square integrable then it admits the spectral representation

$$\Psi(\boldsymbol{x}) = \frac{1}{(2\pi)^d} \int_{\boldsymbol{k} \in R^d} e^{i\boldsymbol{x}\cdot\boldsymbol{k}} d\hat{\Psi}(\boldsymbol{k}) \tag{2.6}$$

with square integrable Fourier spectrum $d\hat{\Psi}(\boldsymbol{k})$

$$\langle \int d\hat{\Psi}_{ij}(\boldsymbol{k}) \int d\hat{\Psi}^*_{ij}(\boldsymbol{k}) \rangle < \infty, \quad \forall i,j \tag{2.7}$$

given by

$$d\hat{\Psi}_{ij}(\boldsymbol{k}) = \frac{-i}{|\boldsymbol{k}|^2} (k_i d\hat{u}_j(\boldsymbol{k}) - d\hat{u}_i(\boldsymbol{k}) k_j) \tag{2.8}$$

where $d\hat{u}_i$ is the Fourier spectrum of the velocity component u_i. The converse is also true (cf. [5]), namely, if (2.7) holds, then (2.6) is stationary and square integrable. By the Cauchy-Schwartz inequality and the Parseval identity, we have

$$
\begin{aligned}
\langle|\tilde{\Psi}_{ij}|^2\rangle &= \langle\int_{\boldsymbol{k}\in R^d} d\hat{\Psi}_{ij}(\boldsymbol{k},\omega)\int_{\boldsymbol{k}'\in R^d} d\hat{\Psi}^*_{ij}(\mathbf{k}',\omega)\rangle \\
&\leq \int\int_{\boldsymbol{k},\boldsymbol{k}'\in R^d}\frac{1}{|\boldsymbol{k}|\cdot|\boldsymbol{k}'|}\left|\langle d\hat{u}_i(\boldsymbol{k},\omega)d\hat{u}_j(\boldsymbol{k}',\omega)\rangle\right| \\
&\leq \sqrt{\int_{\boldsymbol{k}\in R^d}\frac{1}{|\boldsymbol{k}|^2}|\hat{R}_{ij}(\mathbf{k})|d\boldsymbol{k}}\sqrt{\int_{\boldsymbol{k}\in R^d}\frac{1}{|\boldsymbol{k}|^2}|\hat{R}_{jj}(\mathbf{k})|d\boldsymbol{k}}. \qquad (2.9)
\end{aligned}
$$

Thus a sufficient condition for normal diffusion is

$$
\int_{\boldsymbol{k}\in R^d}\frac{1}{|\boldsymbol{k}|^2}|\hat{R}_{ii}(\boldsymbol{k})|d\boldsymbol{k}<\infty,\forall i \qquad (2.10)
$$

where $\hat{R}_{ij}(\boldsymbol{k})$ is Fourier transform of the two-point correlation functions $R_{ij}(\boldsymbol{x})$

$$
R_{ij}(\boldsymbol{x})=\langle u_i(\cdot+\boldsymbol{x})u_j(\cdot)\rangle,\quad \forall i,j. \qquad (2.11)
$$

Condition (2.10) turns out to be sharp. For the precise statements, see [8].

3. Nonstationarity and far-field behavior of stream matrix. The large-scale motion is related to small $\boldsymbol{k}$ behavior of $\hat{R}_{ij}(\boldsymbol{k})$. For $d\geq 3$, $1/|\boldsymbol{k}|^{d+\alpha},\alpha<0$ is locally integrable near $\boldsymbol{k}=0$, therefore (2.10) holds if

$$
\hat{R}_{ij}(\boldsymbol{k})\sim\frac{1}{|\boldsymbol{k}|^{d-2+\alpha}},\quad \text{as } |\boldsymbol{k}|\to 0,\quad \alpha<0 \qquad (3.1)
$$

or, equivalently,

$$
R_{ij}(\boldsymbol{x})\sim\frac{1}{|\boldsymbol{x}|^{2-\alpha}},\quad \text{as } |\boldsymbol{x}|\to\infty,\quad \alpha<0. \qquad (3.2)
$$

In other words, the diffusion is normal if the velocity correlation decays like a power greater than two for $d\geq 3$. Hence anomalous scaling laws for $d\geq 3$ are restricted to the case $\alpha\geq 0$ in (3.2) which, in terms of $\nu=\alpha/2, 0\leq\nu\leq 1$, can be written as

$$
R_{ij}(\boldsymbol{x})\sim\frac{1}{|\boldsymbol{x}|^{2(1-\nu)}},\quad \nu\geq 0. \qquad (3.3)
$$

For $\alpha>2$, a cut-off at large scale $L\gg 1$ is needed to construct a L^2-stationary velocity field

$$
\hat{R}_{ij}(\boldsymbol{k})\sim\begin{cases}\frac{1}{|\boldsymbol{k}|^{\alpha+d-2}}, & |\boldsymbol{k}|\geq 1/L,\quad \alpha>2\\ 0, & |\boldsymbol{k}|<1/L.\end{cases} \qquad (3.4)
$$

The velocity spectrum (3.4) corresponds to the velocity field with the correlation

$$R_{ij}(\boldsymbol{x}) - R_{ij}(0) \sim \begin{cases} |\boldsymbol{x}|^{2(\nu-1)}, & |\boldsymbol{x}| \le L, \quad 1 < \nu = \alpha/2 \le 2 \\ L^{2(\nu-2)}|\boldsymbol{x}|^2, & |\boldsymbol{x}| \le L, \quad 2 < \nu = \alpha/2 \end{cases} \tag{3.5}$$

Note that in the case $\nu > 2$, the asymptotics (3.5) carries a large coefficient of order $L^{2(\nu-2)}$ as can be seen from the following simple calculation.

The Fourier integral

$$R_{ij}(\boldsymbol{x}) - R_{ij}(0) = \frac{1}{(2\pi)^d} \int_{|\boldsymbol{k}| \ge 1/L} (e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - 1)\hat{R}_{ij}(\boldsymbol{k}) d\boldsymbol{k} \tag{3.6}$$

can be split into two parts, the integral over $1/L \le s \le 1/|\boldsymbol{x}|$ and the other one over $s > 1/|\boldsymbol{x}|$. Here the variable $s = \boldsymbol{k}\cdot\boldsymbol{x}/|\boldsymbol{x}|$ is the projection of $\boldsymbol{k}$ in the direction of $\boldsymbol{x}$. The first part can be estimated by expanding the exponential function $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ into power series in $s|\boldsymbol{x}|$. The first order term disappears because of isotropy of $\hat{R}_{ij}$. Thus the leading order contribution from the first part is given by

$$\begin{aligned} & |\boldsymbol{x}|^2 \int_{1/L}^{1/|\boldsymbol{x}|} \int_{R^{d-1}} \frac{s^2}{(s^2 + |\boldsymbol{k}_\perp|^2)^{(\alpha+d)/2-1}} d\boldsymbol{k}_\perp \, ds \\ \sim \; & |\boldsymbol{x}|^2 \int_{1/L}^{1/|\boldsymbol{x}|} \frac{1}{s^{\alpha-3}} ds \int_{R^{d-1}} \frac{1}{(1 + |\boldsymbol{k}'|^2)^{(\alpha+d)/2-1}} d\boldsymbol{k}' \\ \sim \; & |\boldsymbol{x}|^2 (L^{\alpha-4} + |\boldsymbol{x}|^{\alpha-4}). \end{aligned} \tag{3.7}$$

Here the variable $\boldsymbol{k}_\perp$ is the orthogonal projection of $\boldsymbol{k}$ unto the hyperplane perpendicular to $\boldsymbol{x}$ and $\boldsymbol{k}' = \boldsymbol{k}_\perp/s$. Note that the integral $\int_{R^{d-1}} \frac{1}{(1+|\boldsymbol{k}'|^2)^{(\alpha+d)/2-1}} d\boldsymbol{k}'$ converges for $\alpha > 2$.

The second part is of order

$$\begin{aligned} & \int_{1/|\boldsymbol{x}|}^{\infty} \int_{R^{d-1}} \frac{1}{(s^2 + |\boldsymbol{k}_\perp|^2)^{(\alpha+d)/2-1}} d\boldsymbol{k}_\perp \, ds \\ \sim \; & \int_{1/|\boldsymbol{x}|}^{\infty} \frac{1}{s^{\alpha-1}} \int_{R^{d-1}} \frac{1}{(1 + |\boldsymbol{k}'|^2)^{(\alpha+d)/2-1}} d\boldsymbol{k}' \\ \sim \; & |\boldsymbol{x}|^{\alpha-2}. \end{aligned} \tag{3.8}$$

Thus $R_{ij}(\boldsymbol{x}) - R_{ij}(0)$ is of order

$$|\boldsymbol{x}|^2 L^{\alpha-4} + |\boldsymbol{x}|^{\alpha-2}, \quad |\boldsymbol{x}| \le L. \tag{3.9}$$

The first term dominates in the range $|\boldsymbol{x}| \le L$ when $\alpha > 4$.

In two dimensions, $\nu \ge 0$ is only part of the anomalous regime.

The *growth index* μ is a measure of the growth of nonstationary stream matrix in far fields

$$\sqrt{\langle |\Psi(n\boldsymbol{x})|^2 \rangle} \sim n^\mu, \quad \text{as } n \to \infty, \quad \forall |\boldsymbol{x}| \ne 0. \tag{3.10}$$

The index μ for isotropic velocity satisfying (3.3) with $0 < \nu \le 1$ can be calculated exactly:

$$\mu = \nu, \quad \text{for } \nu > 0 \tag{3.11}$$

in dimension $d \ge 2$. For $\nu = 0$, the nonstationary stream matrix grows logarithmically in far fields

$$\langle |\Psi(n\boldsymbol{x})|^2 \rangle \sim \log n, \quad \text{as } n \to \infty. \tag{3.12}$$

The logarithmic growth (3.12) turns out to be generic in two dimensions even if the velocity has fast decaying correlation $R_{ij}(\boldsymbol{x})$, that is, (3.12) holds in two dimensions for $\nu \le 0$. This can be seen as follows.

First, we note that condition (2.10) requires

$$\hat{R}_{ij}(0) = \int_{\boldsymbol{x} \in R^d} R_{ij}(\boldsymbol{x}) d\boldsymbol{x} = 0, \ \forall i, j = 1, 2 \tag{3.13}$$

which is the case for periodic flows but, in general, not for random flows in two dimensions.

Let us consider the case where $\hat{R_{ij}}(\boldsymbol{k})$ is continuous at $\boldsymbol{k} = 0$ (This is the case when $R_{ij}(\boldsymbol{x})$ is absolutely integrable). The variance of the stream function ψ on scale $n \gg 1$ is of the same order as

$$\int_{\frac{1}{n} \le |\boldsymbol{k}| < 1} \frac{1}{|\boldsymbol{k}|^2} d\boldsymbol{k} + \int_{|\boldsymbol{k}| > 1} \frac{1}{|\boldsymbol{k}|^2} |\hat{R}_{ij}(\boldsymbol{k})| d\boldsymbol{k} \tag{3.14}$$

which is of order

$$\log n. \tag{3.15}$$

Thus the logarithmic growth (3.12) holds in the borderline case of (1.40) or (1.41).

4. Scaling exponent by the variational duality argument. It is known that the box diffusivity can be expressed as a pair of minimum principles ([7]). For concreteness, we present the two and three dimensional versions here. First, we have the minimum principle for the upper bound:

$$\sigma_n \boldsymbol{e} \cdot \boldsymbol{e} = \inf_f \int_{[0,1]^d} d\boldsymbol{x} \ (\nabla f \cdot \nabla f + \nabla f' \cdot \nabla f') \tag{4.1}$$

$$\text{with} \quad \Delta f' + \nabla \cdot \Psi_n \nabla f = 0. \tag{4.2}$$

Second, we have the minimum principle for the lower bound in two and three dimensions, respectively:

$$\sigma_n^{-1} \boldsymbol{e} \cdot \boldsymbol{e} = \inf_g \int_{[0,1]^2} \frac{1}{1 + \psi_n^2} (\nabla^\perp g \cdot \nabla^\perp g + \nabla^\perp g' \cdot \nabla^\perp g')$$

$$(4.3) \text{ with } \quad \nabla^\perp \cdot \left[\frac{1}{1+\psi_n^2}\nabla^\perp g'\right] = \nabla^\perp \cdot \left[\frac{\Psi_n}{1+\psi_n^2}\nabla^\perp g\right]$$

$$\sigma_n^{-1}\boldsymbol{e}\cdot\boldsymbol{e} = \inf_{\boldsymbol{G}} \int_{[0,1]^3} (I+\Psi_n\Psi_n^*)^{-1}(\nabla\times\boldsymbol{G}\cdot\nabla\times\boldsymbol{G}+\nabla\times\boldsymbol{G}'\cdot\nabla\times\boldsymbol{G}')$$

$$(4.4) \text{ with } \quad \nabla\times\left[(I+\Psi_n\Psi_n^*)^{-1}\nabla\times\boldsymbol{G}'\right] = \nabla\times\left[(I+\Psi_n\Psi_n^*)^{-1}\Psi_n\nabla\times\boldsymbol{G}\right]$$

Here the trial functions f and g are subject to the boundary conditions whose essential part can be posed as the mean field property

$$\int_{[0,1]^d} \nabla f = \boldsymbol{e} \tag{4.5}$$

$$\int_{[0,1]^2} \nabla^\perp g = \boldsymbol{e} \tag{4.6}$$

$$\int_{[0,1]^3} \nabla\times\boldsymbol{G} = \boldsymbol{e}. \tag{4.7}$$

A good choice of trial functions for the direct and dual principle (4.1) and (4.3) can provide useful upper or lower bounds on σ_n.

4.1. Three dimensions. To derive the anomaly exponent we assume that the velocity statistics has the discrete symmetry of being invariant under the permutations and reflections of the x_i-axes, $i = 1, 2, 3..., d$ and u_i is independent of u_j for $i \neq j$ (Plus, of course, the well-definedness of the exponent). For such flows the velocity correlation can be sumed up in a single function $R(\boldsymbol{x}')$

$$R_{ij}(\boldsymbol{x}) = \delta_{i,j}R(\boldsymbol{x}') \tag{4.8}$$

where $\boldsymbol{x}'$ is the $x_1 \leftrightarrow x_i$ permutation of $\boldsymbol{x}$. With this, it is easy to see that the box diffusivity is a *scalar* asymptotically and hence the exponent is independent of the direction. The discrete symmetry is clearly weaker than the isotropy condition.

From the hypothesis of the existence of the scaling exponent q and (1.20) we know that the direct principle (4.1) satisfies

$$(4.1) \sim n^{2-\frac{1}{\lambda}}. \tag{4.9}$$

What about the dual principle (4.3)? Note that the similarity between the direct and dual principles in the functional forms (except the factor $(I + \Psi_n\Psi_n^*)^{-1}$), the equations defining the nonlocal terms and in the mean field constraints. The mean fields (4.5) and (4.6) have conjugate meanings (most clearly seen in two dimension where $\nabla^\perp$ rotates the mean field direction by $\pi/2$) but the difference would not matter in flows with the full discrete symmetry. Thus the growth index (3.10) (to account for the extra factor $(I+\Psi_n\Psi_n^*)^{-1}$, for $d = 3$, in (4.3)) and (4.9) now suggest that

$$(4.3) \sim cn^{2-\frac{1}{\lambda}-2\mu}. \tag{4.10}$$

It is important to note that (4.3) is a minimum principle thus we expect that the term $(I+\Psi_n\Psi_n^*)^{-1}$ contributes a factor of order

$$\frac{1}{1+\langle|\Psi_n|^2\rangle}\sim n^{-2\mu}\quad\text{but not}\quad |\langle(I+\Psi_n\Psi_n^*)^{-1}\rangle| \tag{4.11}$$

to the minimum of the functional. The latter of (4.11) is often much larger than $n^{-2\mu}$. It is more subtle to estimate the "effective" magnitude of the factor $\frac{1}{1+\psi_n^2}$ in two dimensions. We leave the two dimensions to next section.

We finish the derivation by equating (4.9) with the reciprocal of (4.10), since (4.1) and (4.3) are reciprocal to each other,

$$n^{2-\frac{1}{\lambda}}=(n^{2-\frac{1}{\lambda}-2\mu})^{-1}. \tag{4.12}$$

We have

$$\lambda=\frac{1}{2-\mu}. \tag{4.13}$$

For isotropic flows satisfying (3.3) or (3.4), we obtain the result (1.38) from (3.11).

For the borderline case of (1.40) or (1.41), the logarithmic growth (3.12) holds. Following the same duality argument we have

$$\langle\sigma_n\rangle\sim\sqrt{\log(n)}. \tag{4.14}$$

This concludes the derivation of our results (1.29), (1.31) and (1.38) for three dimensions.

4.2. Two dimensions. Here we discuss the consistency condition which is required for the existence of the exponent q and the validity of formula (1.26) in two dimensions.

Because the stream function ψ_n is an invariant of the flow, the solution for each realization of velocity may be strongly influenced by certain level lines of ψ_n, which may vary greatly from sample to sample, unless the effect of molecular diffusion is sufficient to sample the "typical" or "average" level lines. To put it differently, the streamline configuration, due to rigidity of two dimensional geometry (box-percolating streamlines in orthogonal directions can not coexist), often fluctuate from sample to sample and create artificially anisotropy in a finite box $[0,n]^2$. This is schematically depicted in Fig 1 in which the crossing streamlines from top to bottom edge (solid lines) form channels. The convection-diffusion process would rely on the molecular diffusion to contain the artificial anisotropy. One may compare the effects of convection and diffusion in the following way: The anisotropy is caused by box-percolating streamlines and, for the growth index $\mu>0$, the total width of box-percolating streamlines on a box $[0,n]^2$ is roughly n^μ. Let $d_f\geq 1$ be the fractal dimension of the streamlines.

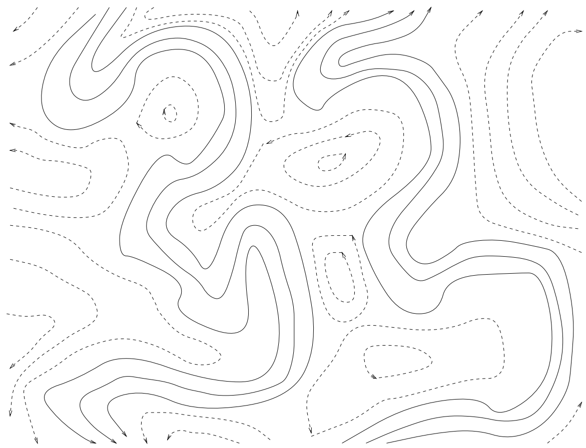

FIG. 1. *Schematic representation of streamlines: crossing streamlines in solid curves*

This mean the typical box-percolating streamlines have the length n^{d_f}. To contain the anisotropy, it requires that the time for particles to transverse the box-percolating streamlines, $n^{2\mu}$, is less than the time to complete the box-percolating streamlines, n^{d_f}, namely,

$$d_f \geq 2\mu. \tag{4.15}$$

The consistency condition (4.15) can be derived more precisely from the variational method as follows.

Let us assume that the probability of a randomly chosen streamline exceeds a diameter n scales like a power law

$$P(n) \sim n^{-\delta}, \quad n \gg 1, \tag{4.16}$$

and the streamlines with diameter greater than n have a length $L(n)$ scaling superlinearly i.e.

$$L(n) \sim n^{d_f}, \quad n \gg 1. \tag{4.17}$$

d_f is the *fractal dimension* of the streamlines and $P(n)$ is the *crossing probability* of streamlines. The exponent δ is related to the fractal dimension d_f and the index μ:

$$n^{-\delta} = \frac{n^{\mu} n^{d_f}}{n^2} \tag{4.18}$$

and hence

$$\delta = 2 - \mu - d_f. \tag{4.19}$$

To have a better sense of the scales we consider the unscaled problem in the large box $[0,n]^2$ and we shall estimate the box diffusivity

$$\sigma_n = \frac{1}{n^2}\int_{[0,n]^2} \nabla\rho\cdot\nabla\rho \tag{4.20}$$

from above and below using the unscaled version of the direct and dual variational principles (4.21) and (4.23) respectively:

$$\sigma_n \boldsymbol{e}\cdot\boldsymbol{e} = \inf_f \frac{1}{n^2}\int_{[0,n]^2} d\boldsymbol{x}\ (\nabla f\cdot\nabla f + \nabla f'\cdot\nabla f') \tag{4.21}$$

$$\text{with}\quad \Delta f' + \nabla\cdot\Psi\nabla f = 0,\quad \text{in } [0,n]^2. \tag{4.22}$$

$$\sigma_n^{-1}\boldsymbol{e}\cdot\boldsymbol{e} = \inf_g \frac{1}{n^2}\int_{[0,n]^2} \frac{1}{1+\psi^2}(\nabla^\perp g\cdot\nabla^\perp g + \nabla^\perp g'\cdot\nabla^\perp g') \tag{4.23}$$

$$\text{with}\quad \nabla^\perp\cdot\frac{1}{1+\psi^2}\nabla^\perp g' = \nabla^\perp\cdot\frac{\Psi}{1+\psi^2}\nabla^\perp g,\quad \text{in } [0,n]^2. \tag{4.24}$$

where the trial functions f and g are subject to the boundary conditions which are essentially the mean field property

$$\frac{1}{n^2}\int_{[0,n]^2} \nabla f = \boldsymbol{e} \tag{4.25}$$

$$\frac{1}{n^2}\int_{[0,n]^2} \nabla^\perp g = \boldsymbol{e}. \tag{4.26}$$

Due to the symmetry of the velocity field the box diffusivity $\sigma_n(\boldsymbol{e})$ should have the same exponent q regardless of the direction $\boldsymbol{e}$ *provided the exponent q is well defined.* Here we see that the problem of artificial anisotropy due to fluctuation in a finite box is closely related to the existence of the exponent q. It turns out that the existence of q implies the consistency condition (4.15). We shall use $\boldsymbol{e} = \boldsymbol{e}_1$ and $\boldsymbol{e} = \boldsymbol{e}_2$ in the direct and the dual principles respectively. Since the velocity field is symmetric in x and y the vertical crossing can not occur for all $n > 0$ and, for the picture (Fig 1) to hold, we need to extract a sequence $n_k \to \infty$, still denoted by n for simplicity.

For the direct principle (4.21) we consider the trial function

$$f(x,y) = \begin{cases} 0, & \text{for } x = 0 \\ n, & \text{for } x = n, \end{cases} \tag{4.27}$$

so that the mean field constraint (4.25) is satisfied, and it takes constant values in the regions separated by the vertically crossing channels, so that its gradient is zero outside the channels. Furthermore, inside the vertically crossing channels, the level sets of the trial function coincide with the streamlines. This is compatible with the boundary condition (4.27) since the channels do not cross either $x = 0$ or $x = n$. With this the nonlocal term in the functional in (4.21) drops out because

$$\nabla \cdot \Psi \nabla f = \boldsymbol{u} \cdot \nabla f = 0. \tag{4.28}$$

The first term in the functional can be estimated by

$$\frac{1}{n^2} \int_{[0,n]^2} \nabla f \cdot \nabla f \leq c \frac{1}{n^2} (\frac{n}{n^\mu})^2 n^{\mu + d_f} = c n^{d_f - \mu} \tag{4.29}$$

in which $\frac{n}{n^\mu}$ is the magnitude of the gradient and $n^{\mu + d_f}$ is the total area of the channels. Thus we have the upper bound

$$\sigma_n \leq c n^{d_f - \mu}. \tag{4.30}$$

To take the same advantage of the flow configuration Fig 1 in the dual problem we take $\boldsymbol{e} = \boldsymbol{e}_2$ in using the dual principle (4.23) to estimate σ_n from below. We take the trial function

$$g = f \tag{4.31}$$

so that the mean field of $\nabla^\perp g$ is in the $\boldsymbol{e}_2$ direction

$$\frac{1}{n^2} \int_{[0,n]^2} \nabla^\perp g = \boldsymbol{e}_2. \tag{4.32}$$

Once again the nonlocal term in the functional of (4.23) drops out because

$$\nabla^\perp \cdot \frac{1}{1 + \psi^2} \Psi \nabla^\perp g = 0. \tag{4.33}$$

The first term in (4.23) can be estimated by

$$\frac{1}{n^2} \int_{[0,n]^2} \frac{1}{1 + \psi^2} \nabla^\perp g \cdot \nabla^\perp g \leq c \frac{1}{n^2} \frac{1}{n^{2\mu}} (\frac{n}{n^\mu})^2 n^{d_f + \mu} = c n^{d_f - 3\mu}. \tag{4.34}$$

Thus we have the lower bound

$$\frac{1}{c} n^{3\mu - d_f} \leq \sigma_n. \tag{4.35}$$

The lower bound (4.35) gives a sufficient condition for super-diffusion:

$$d_f < 3\mu. \tag{4.36}$$

Combining the upper and lower bounds (4.30), (4.35) we get the consistency condition (4.15).

Unfortunately, the fractal dimension d_f is difficult to calculate. One often has to resort to numerical simulation. Two different formulae ([13],[15]) for d_f in terms of μ were proposed but neither of them agrees with the numerical result ($d_f = 1.272$) of [1] for the Manhattan model (see below) for which $\mu = 1/2$.

5. Examples of Flows with Discrete Symmetry.

5.1. Kolmogorov's Spectrum. A widely used model for turbulence velocity spectrum in three dimensions is the modified Kolmogorov spectrum

$$\hat{R}(\boldsymbol{k}) \sim \begin{cases} |\boldsymbol{k}|^{-d-(2+\beta)/3}, & |\boldsymbol{k}| \geq 1/L \\ 0, & |\boldsymbol{k}| < 1/L \end{cases} \tag{5.1}$$

or, equivalently,

$$R(\boldsymbol{x}) \sim |\boldsymbol{x}|^{(2+\beta)/3}, \quad |\boldsymbol{x}| \leq L \tag{5.2}$$

in the inertial range where $0 \leq \beta \leq 3$ accounts for the intermittency effect ([11]). The exponent $\beta = 0$ corresponds to the original 1941 Kolmogorov theory. For the correlation (5.1), the growth index is one with coefficient of order $L^{(2+\beta)/6}$

$$\sqrt{\langle |\Psi_n|^2 \rangle} \sim n L^{(2+\beta)/6}, \quad n \leq L \tag{5.3}$$

which fails to produce Richardson's law, as was noted in the introduction. For $\beta = 0$, (5.3) only resembles Richarson's law, at distance $n \sim L$, in the *order of magnitude*

$$\langle \sigma_n \rangle \sim L^{4/3} \tag{5.4}$$

as $L \to \infty$, but not in its power law relation to n.

5.2. The random composite channel flows. The random composite channel flows is given by the velocity

$$\boldsymbol{u}(\boldsymbol{x}) = (u(y,z), v(x,z), w(x,y)), \quad \boldsymbol{x} = (x,y,z) \tag{5.5}$$

The velocity field given by (5.5) has the discrete symmetry if u, v, w are identically distributed. An explicit example is when u, v, w are independent Bernoulli random variables, taking values ± 1, on the two-dimensional lattice grids orthogonal to x, y, z respectively (see Fig. 2 for the channel flow in the z direction). Because each velocity component depends only on two coordinates, the "effective dimensions" are two.

If the correlation is given by

$$R_{11}(y,z) \sim \frac{1}{(y^2+z^2)^{(1-\nu)}}, \quad \nu > 0, \quad \text{as } y^2+z^2 \to \infty \tag{5.6}$$

then

$$\mu = \nu \tag{5.7}$$

as in (3.11), and

$$\lambda = \frac{1}{2-\nu}. \tag{5.8}$$

For $\nu \leq 0$, the nonstationarity of stream matrix is logarithmic (3.12) and result (4.14) applies.

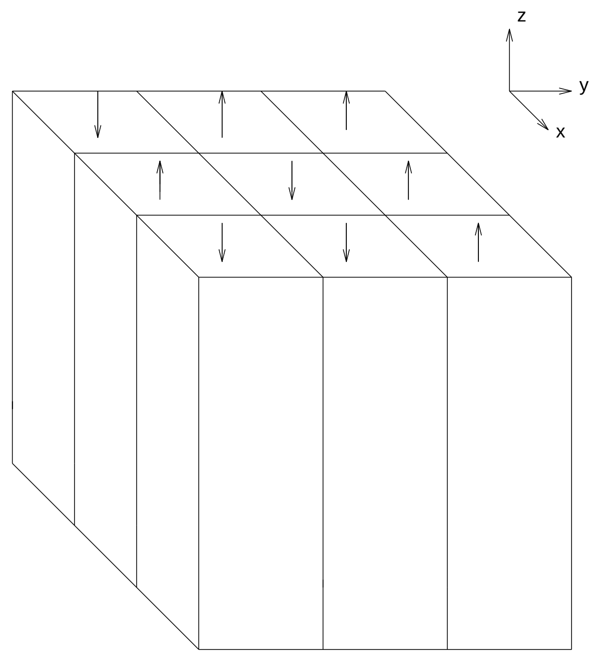

FIG. 2. *A random channel flow in z-direction.*

5.3. The Manhattan model. A two-dimensional analog of (5.5) is the Manhattan model formed by superposing two independent shear layer flows in orthogonal directions

$$\boldsymbol{u}(\boldsymbol{x}) = (u(y), v(x)) \tag{5.9}$$

where $u(y)$ and $v(x)$ are independent, identically distributed stationary random functions with fast decaying correlation and are statistically invariant under the change of sign $u, v \to -u, -v$. The stream function ψ for $\boldsymbol{u}$ is a sum of two independent functions $\psi_1(y), \psi_2(x)$

$$\psi(\boldsymbol{x}) = \psi_1(y) + \psi_2(x) \tag{5.10}$$

with

$$\psi_1(y) = -\int_0^y u(y')dy', \quad \psi_2(x) = \int_0^x v(x')dx'. \tag{5.11}$$

An explicit example is a Poisson construction, where the velocity field is based on a random lattice generated by independent Poisson distributed grid points on both x and y axes. The magnitudes of both x and y components of velocity are constant, say one, in the lattice. The y component

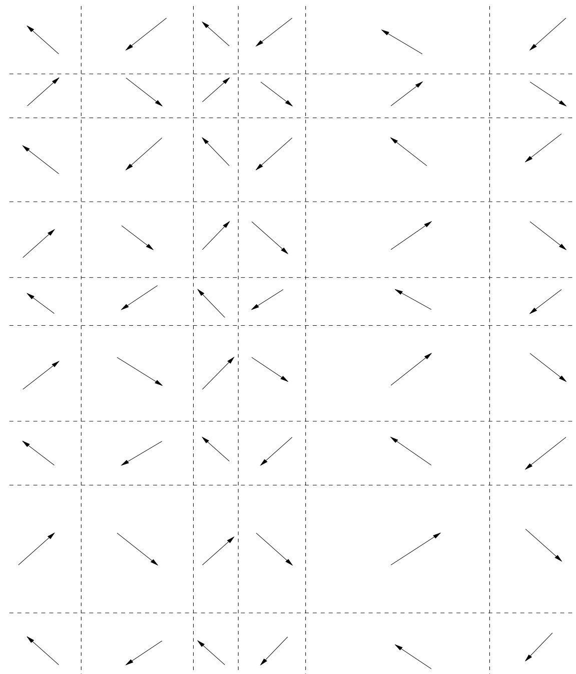

FIG. 3. *The Manhattan Model*

reverses iteself upon a passage from a box to an adjacent box in the x direction with x component remains the same. The x component is similarly determined with the roles of x and y reversed (See Fig. 3).

The velocity field (5.9) has the full discrete symmetry and the growth index is $1/2$ for fast decaying correlation. The numerical value of the fractal dimension d_f in [1] is 1.272 which satisfies the consistency condition (4.15). Thus formula (1.38) yields the result

$$\lambda = \frac{2}{3}. \tag{5.12}$$

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