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Strain Gradient Elasticity for Mode III Cracks: A Hypersingular Integrodifferential Equation Approach

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Abstract

We consider Casal's strain gradient elasticity with two material lengths ℓ, ℓ', associated with volumetric and surface energies, respectively. For a Mode III finite crack we formulate a hypersingular integrodifferential equation for the crack slope supplemented with the natural crack-tip conditions.

The full field solution is then expressed in terms of the crack profile and the Green function, which is obtained explicitly. For $\ell' = 0$, we obtain a closed-form solution for the crack profile. The case of small ℓ' is shown to be a regular perturbation. The question of convergence, as $\ell, \ell' \to 0$, is studied in detail both analytically and numerically.

1 Introduction

Classical elasticity is a scale-free continuum theory in which there is no microstructure associated with material points. In contrast, strain-gradient elasticity enriches the classical continuum with additional material characteristic lengths in order to describe the size (or scale) effects resulting from the underlying microstructures. This consideration generally results in constitutive relations in which the strain energy density W, is not only a function of the classical strain but also the spatial derivatives of deformation, *i.e.* $W = W(\varepsilon, \ell\nabla\varepsilon, \ell^2\nabla^2\varepsilon, ...)$ where ℓ represents generically a material characteristic length. Microstructural size effects can, in theory, be present in any materials: in the case of crystals, the microstructure is the atomic lattice and ℓ is roughly the distance of interaction (Askar 1985; Bardenhagen & Triantafyllidis 1994); in the case of polycrystalline metals or granular materials the microstructure is determined by the compositional grains and probably has a larger characteristic length. In either case, the magnitude of ℓ, after nondimensionalization, represents the ratio of the spatial scale of observation and the scale of the microstructure and is typically small. A graphical way of representing the transition from the classical continuum to the enriched continuum is replacing material *points* in the classical continuum with material *particles* (grains or cells) with internal structure which gives rise to macroscopic effects described by the strain-gradient terms in the constitutive relations. There has been a surge of renewed interest in using strain-gradient theories to account for size (or scale) effects in materials (see, for examples, Lakes 1983, 1986; Smyshlyaev & Fleck 1996; Gao *et al.* 1998; Van Vliet & Van Mier 1999).

Since the pioneering work of Cosserat & Cosserat (1909), various strain-gradient elasticity theories have been proposed and studied by, for example, Toupin (1964), Mindlin (1964), Eringen & Suhubi (1964), Green & Rivlin (1964), Casal (1972), Germain (1973) among others. Toupin-Mindlin's couple-stress theory is isotropic and its simplest kind (the first order strain-gradient theory) has the following strain-energy density given by

$$W = \frac{1}{2}\lambda\epsilon_{ii}\epsilon_{jj} + G\epsilon_{ij}\epsilon_{ji} + \ell^2 G\partial_k\omega_{ij}\partial_k\omega_{ij}, \quad \ell \geq 0 \tag{1}$$
$$\epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad \omega_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i)$$

where G is the shear modulus and λ is the Lamé constant. Here and below we adopt the summation convention that terms are summed over the repeated subscripts. The gradient of the rotation tensor $\partial_k\omega_{ij}$ represents certain curvature-twist effect. Because of the requirement of isotropy, (1) does not have terms representing surface energy (from the boundaries or the crack faces). To include surface energy within the Toupin-Mindlin continuum, one has to go to the second order gradient theory (Mindlin 1965; Wu 1992). On the other hand, Casal's continuum is anisotropic and has the strain-energy density

$$W = \frac{1}{2}\lambda\epsilon_{ii}\epsilon_{jj} + G\epsilon_{ij}\epsilon_{ji} + \ell^2\left(\frac{1}{2}\lambda\partial_k\epsilon_{ii}\partial_k\epsilon_{jj} + G\partial_k\epsilon_{ij}\partial_k\epsilon_{ji}\right) + \ell'\nu_k\partial_k\left(G\epsilon_{ij}\epsilon_{ji} + \frac{1}{2}\lambda\epsilon_{ii}\epsilon_{jj}\right), \tag{2}$$

where ℓ' is another material characteristic length associated with surfaces and ν_k, $\partial_k\nu_k = 0$, is a director field equal to the unit *outer* normal n_k on the boundaries. It is easy to see, after integrating W over the material domain and applying the Stokes theorem, that the term containing ℓ' becomes a surface integral

$$\ell'\left(G\int_{\partial\Omega}\epsilon_{ij}\ \epsilon_{ij}\ dA + \frac{1}{2}\lambda\int_{\partial\Omega}\epsilon_{ii}\ \epsilon_{jj}\ dA\right)$$

corresponding to certain surface energy which is allowed to be negative. Clearly, for $\ell, \ell' \geq 0$, the *total* strain-energy is *positive-definite*. On the other hand, it is straightforward to check that the strain energy *density* (2) is *point-wise* positive for $-1 < \ell'/\ell < 1$. These two facts together imply that, for $\rho = \ell'/\ell > -1$, the *total* energy is positive-definite and the associated boundary-value problems do not have oscillatory solutions. For $\rho < -1$, however, oscillations as well as displacements opposite to the loading condition may arise in the crack profile (see Figures 1, 2 and the discussion in Section 9).

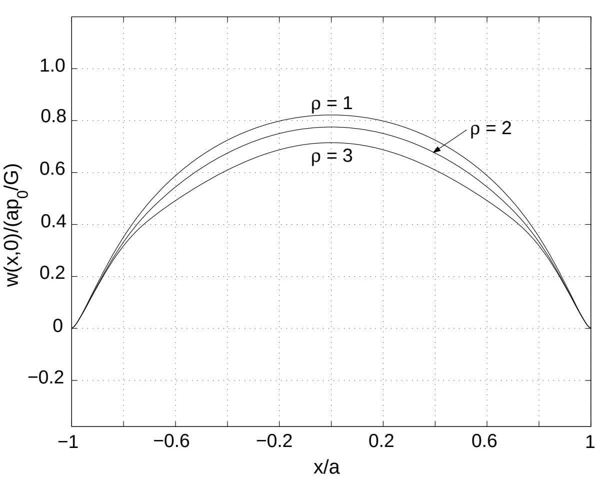

Figure 1: Crack profiles for $\ell = 0.05$ and various $\rho = \ell'/\ell > 1$.

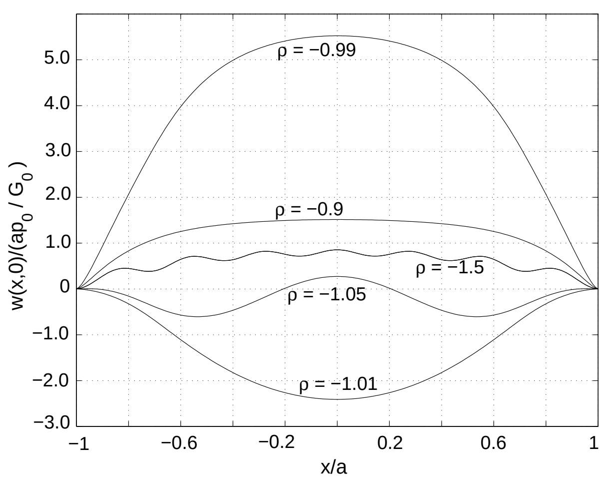

Figure 2: Crack profiles for $\ell = 0.05$ and various $\rho = \ell'/\ell$ near -1.

Recently, Zhang *et al.* (1998) studied the Mode III crack problem for the Toupin-Mindlin continuum with a semi-infinite crack subjected to the classical K_{III} field imposed at the far field or arbitrary anti-plane shear tractions on crack faces. They used the Wiener-Hopf technique of analytic continuation to solve for the solutions and found that the stresses have $r^{-3/2}$ singularity near the crack tip with a (normalized) stress intensity factor significantly larger than the classical one within a zone of size ℓ to the crack tip. Moreover, the crack displacement cusps at the crack tip like $r^{3/2}$, in departure from the classical result of $r^{1/2}$. On the other hand, Exadaktylos *et al.* (1996) and Vardoulakis *et al.* (1996) studied Mode III crack problem with a finite crack in Casal's continuum with or without surface energy term (see (2)) and found a $r^{3/2}$ crack-tip cusping, similar to Zhang *et. al* (1998), but a different stress singularity, $r^{-1/2}$.

To make a fair comparison, let us note that, except for the material length ℓ', Toupin and Mindlin's strain energy and Casal's strain energy for the Mode III crack problem, in which the only nonzero strains are ϵ_{xz} and ϵ_{yz}, can be written in the same form

$$W = 2\left[G(\epsilon_{xz}^2+\epsilon_{yz}^2) + G\ell^2(|\nabla\epsilon_{xz}|^2+|\nabla\epsilon_{yz}|^2) + G\ell'\nu_k\partial_k(\epsilon_{xz}^2+\epsilon_{yz}^2)\right].$$

Namely, for Mode III crack problem, Toupin and Mindlin's continuum is a special case of Casal's continuum with $\ell'=0$. This is not true, of course, for plane crack problems. Besides the presence of the material length ℓ', the boundary conditions used in Vardoulakis *et al.* (1996) and Exadaktylos *et al.* (1996) are somewhat different from, but closely related to, those of Zhang *et al.* (1998) (see Section 3 for details).

One of the main purposes of this article is to resolve the crack-tip asymptotics for a finite crack embedded in an infinite homogeneous medium with anti-plane traction on the crack faces (Mode III) and to study the dependence of solutions on ℓ,ℓ' using the method of hypersingular integrodifferential equations which has been instrumental in studying crack problems in the classical theory (Muskhelishvili 1953; Erdogan *et al.* 1973; Erdogan 1978). *In the special case of $\ell'=0$, the exact solution is obtained in closed form and written in the physical variables such that the crack-tip asymptotics is explicit.* Solutions of Mode I (and mixed-mode) problems for Casal's theory are significantly different from those for Toupin-Mindlin's theory (Sternberg & Muki 1967; Atkinson & Leppington 1977; Xia & Hutchinson 1996) and may be addressed by the method presented in this work.

The other goal of this paper is to answer the question of convergence to the classical linear elastic fracture mechanics, as $\ell,\ell'\to 0$. The results turn out to depend on whether $\rho>-1$ or $\rho<-1$ (Section 8 and 9). We show analytically that the convergence holds for small $\rho\sim 0$ and numerically for $\rho>-1$, and for $\rho\le -1$ the crack profiles diverge as $\ell,\ell'\to 0$. This bifurcation phenomenon is consistent with the above analysis of positive-definiteness of the strain energy.

2 Constitutive relations and equilibrium equations

For the Mode III problem, whose configuration is displayed in Figure 3, the x,y displacements are zero, i.e. $u_x=u_y=0$, and the director field $(\nu_k)=(0,-1,0)$. We set $u_z(x,y)=w(x,y)$.

We define the Cauchy stresses τ_{ij}, the couple stresses μ_{kij} and the total stresses σ_{ij} as:

$$\begin{aligned}
\tau_{ij} = \quad & \partial W/\partial\epsilon_{ij} & = \lambda\delta_{ij}\epsilon_{kk}+2G\epsilon_{ij}+\ell'(\lambda\delta_{ij}\nu_k\partial_k\epsilon_{ll}+2G\nu_k\partial_k\epsilon_{ij})\\
\mu_{kij} = \quad & \partial W/\partial\epsilon_{ij,k} & = \ell^2(\lambda\delta_{ij}\partial_k\epsilon_{ll}+2G\partial_k\epsilon_{ij})+\ell'(\lambda\delta_{ij}\nu_k\epsilon_{ll}+2G\nu_k\epsilon_{ij})\\
\sigma_{ij} = \quad & \tau_{ij}-\partial_k\mu_{kij} & = \lambda\delta_{ij}\epsilon_{kk}+2G\epsilon_{ij}-\ell^2(\lambda\delta_{ij}\nabla^2\epsilon_{ll}+2G\nabla^2\epsilon_{ij})
\end{aligned}$$

and we have, for Mode III crack problem, only the following nonzero stresses:

$$\tau_{xz} \quad = \quad 2G\epsilon_{xz}-2G\ell'\partial_y\epsilon_{xz}$$

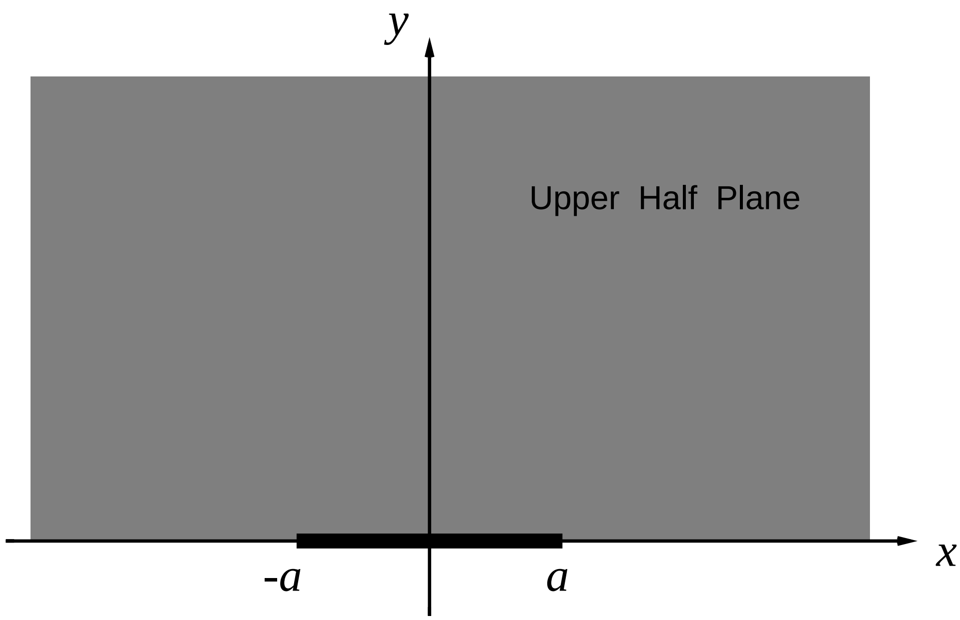

Figure 3: Geometry of the mode III crack problem.

$$
\begin{aligned}
\tau_{yz} &= 2G\epsilon_{yz} - 2G\ell'\partial_y\epsilon_{yz} \\
\mu_{xxz} &= 2G\ell^2\partial\epsilon_{xz}/\partial x \\
\mu_{xyz} &= 2G\ell^2\partial\epsilon_{yz}/\partial x \\
\mu_{yxz} &= 2G(\ell^2\partial\epsilon_{xz}/\partial y - \ell'\epsilon_{xz}) \\
\mu_{yyz} &= 2G(\ell^2\partial\epsilon_{yz}/\partial y - \ell'\epsilon_{yz}) \\
\sigma_{xz} &= 2G(\epsilon_{xz} - \ell^2\nabla^2\epsilon_{xz}) \\
\sigma_{yz} &= 2G(\epsilon_{yz} - \ell^2\nabla^2\epsilon_{yz})
\end{aligned}
$$

A calculation based on the principle of virtual work (Germain 1973) leads to the following equilibrium equations

$$\partial_i\sigma_{ij} + F_j = 0 \tag{3}$$

$$n_i\sigma_{ij} - D_i^t(n_k\mu_{kij}) + (D_l^t n_l)n_k n_i\mu_{kij} + T_j = 0 \tag{4}$$

$$n_k n_i\mu_{kij} + Q_j = 0 \tag{5}$$

for the balance of the external body force F_j, the traction T_j and the double traction Q_j on the boundaries, respectively, where D_i^t stands for the tangential derivatives on the boundaries. For the Mode III problem in the absence of external body force, the equilibrium equation (3) reduces to

$$\partial\sigma_{xz}/\partial x + \partial\sigma_{yz}/\partial y = 0 \ ,$$

which, in the case of homogeneous materials ($G =$ constant.), takes the simple form

$$-\ell^2\nabla^4 w + \nabla^2 w = 0 \quad \text{or} \quad (1 - \ell^2\nabla^2)\nabla^2 w = 0 \ . \tag{6}$$

Equations (4) and (5) become the boundary conditions on the crack faces, which we will discuss next.

3 Boundary conditions

For the convenience of deriving a hypersingular integral equation, we treat the entire x-axis as the boundary on which the boundary conditions are imposed, and the upper-half plane the domain in which equation (6) is to be solved. Naturally the far-field boundary condition is imposed:

$$\lim_{x,y\to\infty} w(x,y) = 0. \tag{7}$$

With the outward unit normal $(n_x, n_y, n_z) = (0, -1, 0)$ on the x-axis, equations (4) and (5) become

$$\begin{aligned} T_z &= \sigma_{yz} - \partial_x \mu_{yyz} && (8)\\ Q_z &= -\mu_{yyz}\ . && (9) \end{aligned}$$

On the crack faces $y = 0, x \in (-a, a)$, the medium is loaded with a nonzero shear traction $T_z = p(x)$ and zero double traction $Q_z = 0$. Thus,

$$\begin{aligned} \sigma_{yz}(x,0) &= \ p(x), \quad |x| < a && (10)\\ \mu_{yyz}(x,0) &= \quad 0, \quad\ \ |x| < a. && (11) \end{aligned}$$

On the ligament $y = 0, |x| > a$, the displacement is assumed to be zero

$$w(x,0) = 0, \quad |x| > a \tag{12}$$

which may be due to one of the following loading conditions. One condition is the antisymmetry of the loading with respect to the x-axis so that the displacement is antisymmetric with respect to the x-axis. In this case, one can furnish another condition

$$\frac{\partial^2 w(x,0)}{\partial y^2} = 0\ , \quad |x| > a. \tag{13}$$

This is the condition used in Zhang *et al.* (1998) in the case of $\ell' = 0$. Or the ligament is clamped to a rigid substrate so that

$$\frac{\partial w(x,0)}{\partial y} = 0\ , \quad |x| > a. \tag{14}$$

The boundary condition studied in much greater detail in the sequel is a linear combination of the above two:

$$-\ell' \epsilon_{yz} + \ell^2 \partial \epsilon_{yz}/\partial y = 0\ , \quad |x| > a. \tag{15}$$

which, in conjunction with (11), implies

$$\mu_{yyz}(x, 0^+) = 0, \quad \forall x \in (-\infty, \infty). \tag{16}$$

This is the condition used in Vardoulakis *et al.* (1996). Conditions (7), (10), (11) together with either (13), (14) or (15) constitute a mixed boundary value problem for equation (6). Conditions (13), (14) and (15) are analogous to the Neumann, Dirichlet and Robin conditions, respectively, in classical potential theory. In the case of $\ell' \neq 0$, conditions (13) and (14) are more complicated to deal with than (15). In the special case $\ell' = 0$, however, condition (15) reduces to condition (13) of Zhang *et al.* (1998).

By standard elliptic PDE theory, the solution $w(x,y)$ of equation (6) with the mixed boundary conditions (10), (11), (12) and (15) (or (13) or (14)) is unique in a general class of functions, and is infinitely differentiable in the interior and continuous up to the boundary. *One of the main goals of this paper is to characterize precisely the behavior of the solution as it approaches the boundary, in particular the points (i.e. the crack-tips) where the boundary conditions change type.*

4 The Green function and integral representation of solution

Let the Fourier transform be defined as

$$\hat{f} = \mathcal{F}(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-ix\xi} dx \ .$$

By the Fourier inversion theorem, we have

$$f = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) \mathrm{e}^{ix\xi} d\xi \ .$$

Let

$$w(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(\xi, y) \mathrm{e}^{ix\xi} d\xi \ ,$$

then from (6), we obtain

$$\ell^2 \frac{d^4 W}{dy^4} - (2\ell^2 \xi^2 + 1) \frac{d^2 W}{dy^2} + (\ell^2 \xi^4 + \xi^2) W = 0 \ . \tag{17}$$

The characteristic equation corresponding to equation (17) is

$$\ell^2 \lambda^4 - (2\ell^2 \xi^2 + 1)\lambda^2 + (\ell^2 \xi^4 + \xi^2) = 0 \ ,$$

which can be factorized as

$$[\ell^2 \lambda^2 - (1 + \ell^2 \xi^2)][\lambda^2 - \xi^2] = 0$$

and solved

$$\lambda = \pm |\xi|, \quad \text{or} \quad \pm \sqrt{\xi^2 + \ell^{-2}}.$$

By the symmetry of the problem, only upper-half plane ($y \geq 0$) is considered, so we keep only the negative roots

$$\lambda_1(\xi) = -|\xi| \quad \text{and} \quad \lambda_2(\xi) = -\sqrt{\xi^2 + \ell^{-2}} \ .$$

The general solution $w(x,y)$ to equation (6) can be given by

$$w(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(\xi) \mathrm{e}^{\lambda_1 y} + B(\xi) \mathrm{e}^{\lambda_2 y} \right] \mathrm{e}^{ix\xi} d\xi, \quad y > 0 \tag{18}$$

satisfies the far field condition (7). The coefficients $A(\xi)$ and $B(\xi)$ are to be determined by the boundary conditions. Note that $\lambda_2(\xi)$ has the following asymptotics

$$\lambda_2(\xi) \quad \sim \quad -|\xi| - \frac{1}{2\ell^2|\xi|} + \frac{1}{8\ell^4|\xi|^3} - \frac{1}{16\ell^6|\xi|^5}, \quad |\xi| \to +\infty \tag{19}$$

which is used in the next section. After substitution, we obtain

$$\begin{aligned} \mu_{yyz}(x,y) \quad &= \quad \frac{G}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big\{ A(\xi) \left(\ell^2 \xi^2 + \ell' |\xi| \right) \mathrm{e}^{-|\xi| y} \\ &+ \quad B(\xi) \left[\ell^2 \xi^2 + 1 + (\ell'/\ell) \sqrt{\xi^2 \ell^2 + 1} \right] \mathrm{e}^{-(y/\ell)\sqrt{\xi^2 \ell^2 + 1}} \Big\} \mathrm{e}^{ix\xi} d\xi \ , \ y > 0 \end{aligned} \tag{20}$$

Condition (16) and equation (20) then imply that

$$B(\xi) = \frac{-\ell^2 \xi^2 - \ell' |\xi|}{(\ell'/\ell)\sqrt{\ell^2 \xi^2 + 1} + \ell^2 \xi^2 + 1} A(\xi) \ . \tag{21}$$

As we will see below, $A(\xi)$ does not decay in ξ for $\ell|\xi| \gg 1$, so Eq. (54) is not well-defined for $y = 0$ and the stress $\sigma_{yz}(x, 0^+)$ should be obtained from Eq. (54) by a limiting procedure $y \to 0^+$ giving rise to Hadamard's finite-part integrals (see the next section and Appendices A, B). On the other hand, the integral in Eq. (18), for $|y| \ll \ell$, is a much nicer object due to cancellation of singularities in A and B. Indeed, from Eq. (21), we see that

$$B(\xi) \sim \left(-1 + \frac{1}{\ell^2\xi^2}\right) A(\xi), \quad \ell|\xi| \gg 1$$

so we have

$$A(\xi) + B(\xi) \sim \frac{A(\xi)}{\ell^2\xi^2}, \quad \ell|\xi| \gg 1.$$

Similar cancellation occurs in Eq. (20) (cf. Eq. (59)-(60)).

Define the slope function

$$\phi(x) = \frac{\partial}{\partial x} w(x, 0^+) \tag{22}$$

so that

$$\phi(x) = 0, \quad |x| > a \quad ,$$

and

$$\int_{-a}^{a} \phi(x)dx = w(a, 0^+) - w(-a, 0^+) = 0 \; . \tag{23}$$

Since (18) implies

$$\frac{\partial w(x,y)}{\partial x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\xi \left[A(\xi)\mathrm{e}^{-|\xi|y} + B(\xi)\mathrm{e}^{-(y/\ell)\sqrt{\ell^2\xi^2+1}}\right] \mathrm{e}^{ix\xi} d\xi \; , \; y \geq 0 \tag{24}$$

we have the integral representation for $\phi(x)$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\xi[A(\xi) + B(\xi)]\mathrm{e}^{ix\xi} d\xi, \quad -\infty < x < \infty$$

Inverting the Fourier transform, we obtain

$$i\xi[A(\xi) + B(\xi)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \phi(t)\mathrm{e}^{-i\xi t} dt := \hat{\phi}(\xi) \quad . \tag{25}$$

Clearly $\hat{\phi}(\xi)/(i\xi) = \hat{w}(\xi, 0^+) := \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} w(t, 0^+)\mathrm{e}^{-i\xi t} dt$.

Substituting (21) into (25), we obtain

$$A(\xi) = \tilde{A}(\xi)\hat{w}(\xi, 0^+) \tag{26}$$
$$B(\xi) = \tilde{B}(\xi)\hat{w}(\xi, 0^+) \tag{27}$$

with

$$\tilde{A}(\xi) = \frac{\rho\sqrt{\ell^2\xi^2 + 1} + \ell^2\xi^2 + 1}{\rho(\sqrt{\ell^2\xi^2 + 1} - \ell|\xi|) + 1}, \quad \rho = \ell'/\ell \tag{28}$$

$$\tilde{B}(\xi) = \frac{\ell^2\xi^2 + \ell'|\xi|}{-\rho(\sqrt{\ell^2\xi^2 + 1} - \ell|\xi|) - 1}, \quad \rho = \ell'/\ell. \tag{29}$$

With (28) and (21), we can rewrite equation (18) as

$$w(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} w(t,0^+)\mathcal{G}(x-t,y)dt \tag{30}$$

where the Green function $\mathcal{G}(x,y)$ is given by

$$\mathcal{G}(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{ix\xi} \left[\tilde{A}(\xi)\mathrm{e}^{-|\xi|y} + \tilde{B}(\xi)\mathrm{e}^{(-y/\ell)\sqrt{\xi^2\ell^2+1}} \right] d\xi \ . \tag{31}$$

In contrast, the displacement $w_c(x,y)$ of the classical elasticity, under the boundary conditions (10), (12) is

$$w_c(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} w_c(t,0^+)\mathcal{G}_c(x-t,y)dt \tag{32}$$

with the Green function

$$\mathcal{G}_c(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-|\xi|y+i(x-t)\xi} d\xi \ dt = \frac{y}{y^2+x^2}. \tag{33}$$

In Section 8 we show the convergence of $\mathcal{G}$ to $\mathcal{G}_c$ as ℓ tends to zero for any $\rho = \ell'/\ell \neq -1$.

5 Hypersingular integrodifferential equations

Substituting (30) in (54), passing to the limit $y \to 0^+$, and using the condition (10), we obtain the following integral equation:

$$\lim_{y\to 0^+} \frac{G}{2\pi} \int_{-a}^{a} \phi(t) \int_{-\infty}^{\infty} \hat{K}(\xi,y)\mathrm{e}^{i\xi(x-t)} d\xi \ dt = -p(x), \qquad |x| < a \tag{34}$$

with the kernel

$$\hat{K}(\xi,y) = \frac{|\xi|}{i\xi} \frac{(\ell'/\ell)\sqrt{\ell^2\xi^2+1} + \ell^2\xi^2 + 1}{(\ell'/\ell)\sqrt{\ell^2\xi^2+1} - \ell'|\xi| + 1} \mathrm{e}^{-|\xi|y} \quad .$$

The limit $y \to 0^+$ in (34) is singular since $\hat{K}(\xi,0)$ does not decay in ξ. So we write

$$\hat{K}(\xi,0) = \hat{K}_\infty(\xi) + \hat{K}_0(\xi)$$

with the nondecaying part $\hat{K}_\infty(\xi,0)$ given by

$$\hat{K}_\infty(\xi,0) = \frac{|\xi|}{i\xi} \left[1 - \frac{1}{4}\left(\frac{\ell'}{\ell}\right)^2 + \frac{\ell'}{2}|\xi| + \ell^2\xi^2 \right] \tag{35}$$

and the decaying part $\hat{K}_0(\xi)$ given by

$$\begin{aligned} \hat{K}_0(\xi) &= \frac{|\xi|}{i\xi} \frac{[\ell'|\xi|/2 + (\ell'/2\ell)^2]\left(\sqrt{\ell^2\xi^2+1} - \ell|\xi|\right) + (\ell'/\ell)^3/4}{\ell'/\ell + \sqrt{\ell^2\xi^2+1} + \ell|\xi|} \\ &= \rho\frac{|\xi|}{i\xi} \frac{(\ell|\xi|/2 + \rho/4)\left(\sqrt{\ell^2\xi^2+1} - \ell|\xi|\right) + \rho^2/4}{\rho + \sqrt{\ell^2\xi^2+1} + \ell|\xi|}, \quad \rho = \ell'/\ell. \end{aligned} \tag{36}$$

By (35) and the results of Appendix B

$$\int_{-\infty}^{\infty} \hat{K}_\infty(\xi,y)\mathrm{e}^{i\xi(x-t)} d\xi$$

converges, as $y \to 0^+$, in the sense of distribution, to the hypersingular kernels of the following equation (37) whereas $\hat{K}_0(\xi)$ gives rise to the regular kernel K_0. Thus, as $y \to 0^+$, equation (34) becomes

$$-\frac{2\ell^2}{\pi} \oint_{-a}^{a} \frac{\phi(t)}{(t-x)^3} dt + \frac{1-\rho^2/4}{\pi} \fint_{-a}^{a} \frac{\phi(t)}{t-x} dt + \frac{1}{\pi} \int_{-a}^{a} K_0(t-x)\phi(t)dt - \frac{\ell'}{2}\phi'(x) = \frac{p(x)}{G}, \quad (37)$$

where $|x| < a$, and the regular kernel K_0 can be written as

$$K_0(t-x) = 2\int_0^\infty \hat{K}_0(\xi) \sin[\xi(t-x)] d\xi \quad (38)$$

in view of the anti-symmetry of $\hat{K}_0(\xi)$. Here $\oint_{-a}^{a}$ denotes Hadamard's finite-part integral and $\fint_{-a}^{a}$ denotes Cauchy's principal value integral (Folland 1992; Meyer 1998). Since the dominant kernel in (37) is cubically singular, we need to furnish, in addition to (23), two more crack-tip conditions:

$$\phi(a) = \phi(-a) = 0 \quad (39)$$

in departure from the classical elasticity in which the displacement gradient $\phi(x)$ has the end-point asymptotics

$$\phi(x) = O(\frac{1}{\sqrt{a^2 - x^2}}), \quad \text{as } x \to a^-, \ (-a)^+ \quad (40)$$

(see below for more discussion on this). As we shall see, a much weaker condition than (39) is sufficient to ensure the uniqueness of solution which, in turn, can be shown to satisfy (39).

An important observation is that once $\phi(x)$ is solved from Eq. (37) the coefficients $A(\xi)$ and $B(\xi)$ can be obtained from (28) and (29), respectively, and, then, the full field solution $w(x, y)$ is explicitly given by (18).

In contrast to condition (15) which is assumed in the preceding discussion, it is more convenient to use two density functions, ϕ, ψ for (13) or (14) and the resulting system becomes two coupled hypersingular integrodifferential equations:

Condition (13): For $|x| < a$,

$$\begin{aligned} -\frac{2\ell^2}{\pi} \oint_{-a}^{a} \frac{\phi(t)}{(t-x)^3} dt + \frac{1}{\pi} \fint_{-a}^{a} \frac{\phi(t)}{t-x} dt - \frac{\ell^2}{\pi} \fint_{-a}^{a} \frac{\psi(t)}{t-x}\, dt &= p(x)/G \\ \frac{\ell'}{4\pi} \fint_{-a}^{a} \frac{\phi(t)}{t-x} dt - \frac{\ell'}{\pi} \int_{-a}^{a} k_1(t-x)\phi(t)dt + \frac{1}{\pi} \int_{-a}^{a} k_2(t-x)\psi(t)\, dt &= 0 \end{aligned}$$

with the density functions

$$\phi(x) = \frac{\partial}{\partial x} w(x, 0^+), \qquad \psi(x) = \frac{\partial}{\partial x} \left[\frac{\partial^2 w(x,0)}{\partial y^2} \right]$$

and regular kernels k_1, k_2.

Condition (14): For $|x| < a$,

$$\begin{aligned} \frac{-4\ell^2}{\pi} \oint_{-a}^{a} \frac{\phi(t)}{(t-x)^3} dt + \frac{3}{2\pi} \fint_{-a}^{a} \frac{\phi(t)}{t-x} dt + \frac{1}{\pi} \int_{-a}^{a} k_3(t-x)\phi(t)dt - 2\ell^2\psi'(x) & \\ + \frac{1}{\pi} \int_{-a}^{a} k_4(t-x)\psi(t)dt &= p(x)/G \\ -\ell^2\phi'(x) + \frac{1}{\pi} \int_{-a}^{a} k_5(t-x)\phi(t)dt + \frac{2\ell^2}{\pi} \fint_{-a}^{a} \frac{\psi(t)}{t-x} dt + \frac{1}{\pi} \int_{-a}^{a} k_6(t-x)\psi(t)dt &= 0 \end{aligned}$$

with the density functions

$$\phi(x) = \frac{\partial}{\partial x} w(x, 0^+), \qquad \psi(x) = \frac{\partial^2 w(x,0)}{\partial x \partial y}$$

and the regular kernels k_3, k_4, k_5, k_6.

In addition, appropriate crack-tip conditions also need to be specified to ensure existence and uniqueness of solution. The above two systems of equations are more difficult to analyse and will be studied in a separate investigation.

6 Solutions of the integral equations

It is convenient to nondimensionalize equation (37) by the half crack length a. In view of the fact that both $\phi(x)$ and $p(x)/G$ are dimensionless, this amounts to normalizing the variables by a in the equation and replacing ℓ, ℓ' by $\tilde{\ell} = \ell/a, \tilde{\ell}' = \ell'/a$, respectively. But we will continue to call $\tilde{\ell}$ by ℓ and $\tilde{\ell}'$ by ℓ' for ease of notation.

6.1 Case $\ell' = 0$: closed form solution

Note that the regular kernel $K_0(t-x)$ in equation (37) has a factor ℓ', so it drops out from the equation when $\ell' = 0$.

After normalizing by the half crack length a, equation (37) becomes:

$$-\frac{2\ell^2}{\pi} \,\,=\!\!\!\!\!\!\!\int_{-1}^{1} \frac{\phi(t)}{(t-x)^3} dt + \frac{1}{\pi} \,-\!\!\!\!\!\!\int_{-1}^{1} \frac{1}{t-x} \phi(t) dt = p(x)/G \ , \quad |x| < 1 \ . \tag{41}$$

Let H denote the finite Hilbert transform

$$\mathsf{H}[\phi](x) = \frac{1}{\pi} \,-\!\!\!\!\!\!\int_{-1}^{1} \frac{\phi(t)}{t-x} dt \ .$$

Then, by the definition of Hadamard's finite-part integrals (Appendix A), equation (41) is a second order differential equation for $\mathsf{H}[\phi](x)$:

$$-\ell^2 \mathsf{H}[\phi]''(x) + \mathsf{H}[\phi](x) = p(x)/G \ ,$$

which has the general solution

$$\mathsf{H}[\phi](x) = -\frac{1}{\ell^2}\, \mathrm{e}^{x/\ell} \int_{-1}^{x} \left[\mathrm{e}^{-2s/\ell} \int_{-1}^{s} \mathrm{e}^{t/\ell}\, \frac{p(t)}{G}\, dt \right] ds + C_1 \mathrm{e}^{x/\ell} + C_2 \mathrm{e}^{-x/\ell} \ . \tag{42}$$

Set

$$f(x) = -\frac{1}{\ell^2}\, \mathrm{e}^{x/\ell} \int_{-1}^{x} \left[\mathrm{e}^{-2s/\ell} \int_{-1}^{s} \mathrm{e}^{t/\ell}\, \frac{p(t)}{G}\, dt \right] ds \ ,$$

then we have

$$\frac{1}{\pi} \,-\!\!\!\!\!\!\int_{-1}^{1} \frac{1}{t-x} \phi(t) dt = f(x) + C_1 \mathrm{e}^{x/\ell} + C_2 \mathrm{e}^{-x/\ell} \equiv g(x) \ . \tag{43}$$

It is well known (Tricomi 1957) that the solution $\phi(x)$ of equation (43), with condition (23), is unique in $L^p[-1,1]$ for any $p > 1$, where $L^p[-1,1]$ is defined by

$$L^p[-1,1] = \left\{ f : [-1,1] \to \mathcal{R} \ \mid \ \|f\|_p = \left[\int_{-1}^{1} |f(x)|^p dx \right]^{1/p} < \infty \right\} \ ,$$

and $\phi(x)$ can be written as

$$\begin{aligned}\phi(x) &= \frac{\sqrt{1-x^2}}{\pi}\fint_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}(x-t)}dt+\frac{x}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt \\ &+\frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{tg(t)}{\sqrt{1-t^2}}dt+\frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\phi(t)dt \end{aligned}\tag{44}$$

provided that Eq. (44) is well-defined. For this, it suffices, for example, that $g(x)\in L^p[-1,1]$ for some $p>2$, so $g(t)/\sqrt{1-t^2}\in L^{1+}[-1,1]$.

Under condition (23) and a stronger integrability condition, $\phi\in L^p[-1,1]$ for some $p>2$ (instead of condition (39)), we then have the conditions determining $g(x)$

$$\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt=0,\quad \int_{-1}^{1}\frac{tg(t)}{\sqrt{1-t^2}}dt=0$$

or equivalently

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt+C_1\int_{-1}^{1}\frac{\mathrm{e}^{t/\ell}}{\sqrt{1-t^2}}dt+C_2\int_{-1}^{1}\frac{\mathrm{e}^{-t/\ell}}{\sqrt{1-t^2}}dt = 0 \tag{45}$$

$$\int_{-1}^{1}\frac{tf(t)}{\sqrt{1-t^2}}dt+C_1\int_{-1}^{1}\frac{t\mathrm{e}^{t/\ell}}{\sqrt{1-t^2}}dt+C_2\int_{-1}^{1}\frac{t\mathrm{e}^{-t/\ell}}{\sqrt{1-t^2}}dt = 0 \tag{46}$$

which determine uniquely the constants C_1, C_2:

$$\begin{aligned} C_1 &= -\left(2\int_{-1}^{1}\frac{\mathrm{e}^{t/\ell}}{\sqrt{1-t^2}}dt\right)^{-1}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt-\left(2\int_{-1}^{1}\frac{t\mathrm{e}^{t/\ell}}{\sqrt{1-t^2}}dt\right)^{-1}\int_{-1}^{1}\frac{tf(t)}{\sqrt{1-t^2}}dt \\ C_2 &= -\left(2\int_{-1}^{1}\frac{\mathrm{e}^{t/\ell}}{\sqrt{1-t^2}}dt\right)^{-1}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt+\left(2\int_{-1}^{1}\frac{t\mathrm{e}^{t/\ell}}{\sqrt{1-t^2}}dt\right)^{-1}\int_{-1}^{1}\frac{tf(t)}{\sqrt{1-t^2}}dt. \end{aligned}$$

With the above proviso, equation (44) becomes

$$\phi(x)=\frac{\sqrt{1-x^2}}{\pi}\fint_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}(x-t)}dt. \tag{47}$$

While the form (47) makes explicit the crack-tip asymptotics $O(\sqrt{1-x^2})$ for the slope $\phi(x)$, the following alternative form [27] is also useful for analyzing the limiting behavior as $\ell\to 0$:

$$\phi(x)=\frac{1}{\pi\sqrt{1-x^2}}\left[\fint_{-1}^{1}\frac{\sqrt{1-t^2}\,f(t)}{x-t}dt+C_1\fint_{-1}^{1}\frac{\sqrt{1-t^2}\,\mathrm{e}^{t/\ell}}{x-t}dt+C_2\fint_{-1}^{1}\frac{\sqrt{1-t^2}\,\mathrm{e}^{-t/\ell}}{x-t}dt\right] \tag{48}$$

since the limit has the singularity like $(\sqrt{1-x^2})^{-1}$ near the crack-tips (see Section 8, Eq. (66)). To be consistent with the expression (47), the apparent singularity in (48) must be canceled.

The unique solution satisfying (39) corresponds to the following choice of C_1, C_2. First we note that, for $f(x)\in L^p[-1,1], p>2$,

$$\mathsf{H}[\sqrt{1-t^2}f](-1)=\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\,f(t)dt<\infty$$

$$\mathsf{H}[\sqrt{1-t^2}f](1)=-\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\,f(t)dt<\infty$$

$$\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{-t/\ell}](-1) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\ \mathrm{e}^{-t/\ell}dt = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\ \mathrm{e}^{t/\ell}dt = -\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{t/\ell}](1) < \infty$$

$$-\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{-t/\ell}](1) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\ \mathrm{e}^{-t/\ell}dt = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\ \mathrm{e}^{t/\ell}dt = \mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{t/\ell}](-1) < \infty$$

Thus, in the presence of the factor $1/\sqrt{1-x^2}$ in equation (48), the constants C_1 and C_2 must satisfy

$$\mathsf{H}[\sqrt{1-t^2}f](-1) + C_1\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{t/\ell}](-1) + C_2\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{-t/\ell}](-1) = 0 \tag{49}$$

$$\mathsf{H}[\sqrt{1-t^2}f](1) + C_1\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{t/\ell}](1) + C_2\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{-t/\ell}](1) = 0. \tag{50}$$

The determinant of the above system is

$$\begin{aligned}
&\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{t/\ell}](-1)\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{-t/\ell}](1) - \mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{-t/\ell}](-1)\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{t/\ell}](1)\\
&= \left\{\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{t/\ell}](-1)\right\}^2 - \left\{\mathsf{H}[\sqrt{1-t^2}\mathrm{e}^{t/\ell}](1)\right\}^2\\
&\neq 0
\end{aligned}$$

so C_1 and C_2 are uniquely determined by (49)-(50). It can be shown directly that with this choice of C_1, C_2, Eq. (48) has the crack-tip asymptotics $O(\sqrt{1-x^2})$. The idea is that the expression

$$\int_{-1}^{1}\frac{\sqrt{1-t^2}\ f(t)}{t-x}dt + C_1\int_{-1}^{1}\frac{\sqrt{1-t^2}\ \mathrm{e}^{t/\ell}}{t-x}dt + C_2\int_{-1}^{1}\frac{\sqrt{1-t^2}\ \mathrm{e}^{-t/\ell}}{t-x}dt$$

generally has the asymptotics $O(1-x^2)$ near the crack tips $x=\pm1$. We leave the details to the reader.

6.2 Case $\ell' \neq 0$: regular perturbation

Integrating eq. (37) once in x, we obtain

$$\begin{aligned}
-\frac{\ell^2}{\pi}\int_{-1}^{1}\frac{\phi(t)}{(t-x)^2}dt + \frac{1-\rho^2/4}{\pi}\int_{-1}^{1}\log|t-x|\phi(t)dt + \frac{1}{\pi}\int_{-1}^{1}\tilde{K}_0(t-x)\phi(t)dt - \frac{\ell'}{2}\phi(x)\\
= \int_0^x p(t)/G dt + C_0\ , \quad |x|<1
\end{aligned} \tag{51}$$

where $\tilde{K}_0(t)$ is a primitive function of the regular kernel K_0: $\tilde{K}_0'(t) = K_0(t)$. The constant C_0 is to be determined by the condition (23). With condition (39), eq. (51) is a type of quadratically singular integral equation, studied in Martin (1991), in which the end-point asymptotics of $\phi(x)$ was proved to be $O(\sqrt{1-x^2})$ by using the Mellin transform.

The crack-tip asymptotics can also be derived in another way. Integrating equation (37) twice in x, we obtain

$$\begin{aligned}
-\ell^2\mathsf{H}[\phi](x) \quad &+ \quad \frac{1-\rho^2/4}{\pi}\int_{-1}^{1}\int_{-1}^{x}\log|t-s|ds\phi(t)dt + \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{x}ds\int_{-1}^{s}d\sigma K_0(t-\sigma)\phi(t)dt\\
&- \quad \frac{\ell'}{2}\int_{-1}^{x}\phi(t)dt \ = \ \frac{1}{G}\int_{-1}^{x}ds\int_{-1}^{s}d\sigma p(\sigma) + C_1x + C_0\ ,
\end{aligned}$$

which is a generalized Cauchy singular integral equation

$$-\ell^2\mathsf{H}[\phi](x) + \int_{-1}^{1}K(x,t)\phi(t)dt = \int_{-1}^{x}ds\int_{-1}^{s}d\sigma p(\sigma)/G + C_1x + C_0 \tag{52}$$

with a regular kernel

$$K(x,t) = \frac{1-(\ell'/2\ell)^2}{\pi}\int_{-1}^{x}\log|t-s|ds + \frac{1}{\pi}\int_{-1}^{x}ds\int_{-1}^{s}d\sigma K_0(t-\sigma) - \frac{\ell'}{2}\mathbf{I}_{[-1,\ x]}(t)\ ,$$

where $\mathbf{I}_{[-1,\ x]}$ is the characteristic function of the interval $[-1,\ x]$, $\forall|x|<1$, i.e.

$$\mathbf{I}_{[-1,\ x]}(t) = \begin{cases} 1\ , & \text{if } \ t\in[-1,\ x]\ , \\ 0\ , & \text{if } \ t\notin[-1,\ x]\ . \end{cases}$$

Since

$$\int_{-1}^{1}\int_{-1}^{1}K^2(x,t)dtdx < \infty\ ,$$

the integral operator

$$\mathsf{K}[\phi](x) \equiv \int_{-1}^{1}K(x,t)\phi(t)dt$$

is a Hilbert-Schmidt operator on $L^2[-1,1]$. Therefore the solution ϕ has the same end-point asymptotics as that of the solutions $\tilde{\phi}$ of the dominant equation

$$-\ell^2\mathsf{H}[\phi](x) = f(x) + C_1x + C_0 \tag{53}$$

subject to the same set of end-point conditions (Muskhelishvili, 1953). The end-point asymptotics of the solution of Eq. (53) can be analyzed as before. We will not repeat it here.

7 Stress asymptotics ahead of crack-tips

In this section we recover the original length unit so the crack length is $2a$ and the slope function is $\phi(x/a), x\in(-a,a)$ where $\phi(t), t\in(-1,1)$ is the solution of Eq. (37).

The full-field stress is given by

$$\sigma_{yz}(x,y) \quad = \quad -\frac{G}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\xi|\tilde{A}(\xi)\hat{w}(\xi,0^+)\mathrm{e}^{-|\xi|y+ix\xi}d\xi\ , \quad y>0 \tag{54}$$

which is analogous to its classical counterpart

$$\sigma_{yz}(x,y) \quad = \quad -\frac{G}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\xi|\tilde{A}_c(\xi)\hat{w}_c(\xi,0^+)\mathrm{e}^{-|\xi|y+ix\xi}d\xi\ , \quad y>0$$

with $\tilde{A}_c(\xi)=1$. In the sequel, we focus on the stress along the ligament which has the alternative expression given by the left hand side of Eq. (37).

First let us analyze the asymptotics of $\mathsf{H}[\phi](z)$ as $z\to a^+$. We write

$$\phi(t/a) = \sqrt{1-t^2/a^2}\,u(t/a)$$

and we know that $u(\pm1)\neq 0$ in general. Set

$$z = a(1+\epsilon)\ , \quad 0<\epsilon\ll 1$$

A simple calculation yields

$$
\begin{aligned}
\mathsf{H}[\phi](1+\epsilon) &= -\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}u(t)dt + \frac{a\epsilon}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}u(t)}{(1-t)(1+\epsilon-t)}dt \\
&= \phi(1) + \frac{a\epsilon}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{u(t)}{1+\epsilon-t}dt \\
&\sim \frac{2\sqrt{2}u(1^-)}{\pi}a\sqrt{\epsilon}\int_{0}^{\infty}\frac{d\tau}{\tau^2+1}\ , \quad \text{as } \ \epsilon \to 0\ ,
\end{aligned}
$$

since $\phi(1) = 0$. On the other hand, for the regular kernel $K(x,t)$ in Eq. (52) the function

$$
F(z/a) \equiv \int_{-1}^{1} K(z/a,t)\phi(t)dt
$$

is twice continuously differentiable and its second derivative generally has a finite limit

$$
\lim_{z\to a^+} F''(z) < \infty\ . \tag{55}
$$

Thus the second anti-derivative of the stress ahead of $x = 1$ has the asymptotics

$$
\int_a^x dt\int_a^t \sigma_{yz}(\tau)d\tau = O(\sqrt{x/a-1}) \quad \text{as} \ \ x \to a^+\ .
$$

In other words,

$$
\sigma_{yz}(x/a,0) \sim \frac{\tilde{K}_{III}\ell/a}{(x/a-1)^{3/2}} = \frac{K_{III}\ell}{\sqrt{2\pi}(x-a)^{3/2}} \quad \text{as} \ \ x \to a^+\ ,
$$

with the mode III stress intensity factor (SIF)

$$
K_{III} = \sqrt{2\pi}\sqrt{a}\tilde{K}_{III}
$$

where $\tilde{K}_{III}$ is the normalized SIF, independent of the crack length. The SIF K_{III} of the gradient elasticity is defined so as to have the same unit as its counterpart in the classical elasticity. It should be noted also, because of (55), the other singular term $O\left((x/a-1)^{-1/2}\right)$ in the asymptotic expansion of $\sigma_{yz}(x)$ as $x \to a^+$ is also determined by the dominant cubically singular kernel. In Table 1, we see that the numerical values of K_{III} converge to the *negative* classical value of SIF (i.e. -1) for $\rho > -1$ and diverge for $\rho < -1$ as $\ell, \ell' \to 0$.

8 Convergence to classical elasticity as $\ell, \ell' \to 0$

On one hand, it is natural to expect the convergence of the gradient elasticity to the classical elasticity in the limit $\ell, \ell' \to 0$ in a suitable sense; on the other hand, the convergence can not be *uniform* throughout the domain for certain physical quantities in view of the fact that strains and stresses of the gradient elasticity have different kinds of asymptotics near the crack-tips from those of the classical elasticity as we have shown in the preceding sections. In this section, we show analytically the convergence results for small ρ and, in the next section, we show numerically the convergence results for $\rho > -1$ and the divergence results for $\rho \le -1$.

8.1 Convergence of the Green functions

We consider a general form of the classical limit in which ℓ , ℓ' tend to zero with a finite ratio ρ

$$\lim_{\ell,\ell'\to 0} \ell'/\ell = \rho. \tag{56}$$

First we analyze the asymptotic behaviors of $\tilde{A}(\xi), \tilde{B}(\xi)$ as given by (28), (29). We divide the domain into two regions: $\ell|\xi| \ll 1$ and $\ell|\xi| \gg 1$. Clearly, we have, for $\ell|\xi| \ll 1$,

$$\tilde{A}(\xi) \sim \begin{cases} 1+\ell^2\xi^2/(2-2\rho), & \text{if } \rho \neq -1 \\ \ell|\xi|/2, & \text{if } \rho = -1. \end{cases} \tag{57}$$

$$\tilde{B}(\xi) \sim \begin{cases} -\ell'|\xi|/(\rho+1), & \text{if } \rho \neq -1 \\ 1, & \text{if } \rho = -1 \end{cases} \tag{58}$$

and, for $\ell|\xi| \gg 1$,

$$\tilde{A}(\xi) \sim \ell^2\xi^2 + \ell'|\xi|/2 + 1 - \rho^2/4 \tag{59}$$

$$\tilde{B}(\xi) \sim -\ell^2\xi^2 - \ell'|\xi|/2 + \rho^2/4. \tag{60}$$

In Eq. (59)-(60) we write several leading terms of the asymptotic expansion because they are related to the cancellation of singularities alluded to in the discussion after Eq. (21). As a result of this cancellation, we have from Eq. (59)-(60) that

$$\hat{\mathcal{G}}(\xi,y) \sim \mathrm{e}^{-y|\xi|} \qquad \text{for } \ell|\xi| \gg 1 \tag{61}$$

after a simple calculation taking into account of the exponential factors in (31). This asymptotics (61) shows the absence of boundary layer behavior (i.e. $y/\ell \ll 1$) in Green's functions as $\ell, \ell' \to 0$. Outside the $\ell-$neighborhood of the $x-$axis (i.e. $y/\ell \gg 1$), Green's function is dominated by the contribution from $\tilde{A}(\xi)$, $|\xi|\ell \ll 1$ due to the much smaller exponential factor associated with $\tilde{B}(\xi)$ in (31), so, again, we have

$$\hat{\mathcal{G}}(\xi,y) \sim \mathrm{e}^{-y|\xi|}\tilde{A}(\xi) \sim \mathrm{e}^{-y|\xi|} \qquad \text{for } \ell|\xi| \ll 1, \tag{62}$$

except for the special case $\rho = -1$, for which case

$$\hat{\mathcal{G}}(\xi,y) \sim \mathrm{e}^{-y|\xi|}\tilde{A}(\xi) \sim \mathrm{e}^{-y|\xi|}\ell|\xi|/2 \qquad \text{for } \ell|\xi| \ll 1. \tag{63}$$

Eq. (61) and (62) shows the convergence of Green's functions to the classical one throughout the domain; while Eq. (63) clearly shows the *divergence* Green's functions for $\rho = -1$ in the region outside the $\ell-$neighborhood of the $x-$axis, i.e. $y \gg \ell$ (cf. (32)).

Therefore the *full-field* convergence of the displacement (30) to the classical one requires only the *uniform* convergence of the crack displacement $w(x,0)$, as determined from eq. (37), to that of the classical elasticity on $[-1,1]$. This is addressed in the next section. The *derivatives* of the displacement (such as strains and stresses), however, may still develop different singularities in the ℓ-neighborhood of the *crack-tips* preventing their uniform convergence (see Figures 4 and 5).

8.2 Convergence of crack displacement

Following from the above asymptotics, the regular kernel K_0 has a singular limit as $\ell, \ell' \to 0$

$$\lim_{\ell,\ell'\to 0} \frac{1}{\pi}\int_{-1}^{1} K_0(t-x)\phi(t)dt = \frac{\rho^2}{4\pi}\int_{-1}^{1} \frac{\phi(t)}{t-x}dt$$

since

$$\lim_{\ell,\ell'\to 0} K_0(\xi) = \frac{|\xi|}{i\xi}\frac{\rho^2}{4}.$$

Thus in the limit (56) the Cauchy singular integral equation of the classical elasticity (Muskhelishvili, 1953) is formally recovered from (37)

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\phi(t)}{t-x}dt = p(x)/G, \quad |x|<1. \tag{64}$$

In the following, we show analytically the convergence to the classical elasticity of the separate limits: $\lim_{\ell\to 0}\lim_{\ell'\to 0}$ and, for the more complicated case of simultaneous limit (56), we show some numerical results (Figures 4 and 5 for the convergence of the slopes and the stresses for $\rho > -1$; Figure 6 for the divergence results for $\rho < -1$; Table 1 for the stress singularity factors).

In view of the integrated form of Eq. (52), the first limit of $\ell' \to 0$ with fixed ℓ is a regular perturbation by a vanishing Hilbert-Schmidt operator of

$$-\ell^2\, \mathsf{H}[\phi](x) + \frac{1}{\pi}\int_{-1}^{1}\phi(t)\int_{-1}^{x}\log|t-s|ds\,dt = \int_{-1}^{x}ds\int_{-1}^{s}d\sigma\; p(\sigma)/G + C_1 x + C_0 \tag{65}$$

as noted in Section 6.2. So the convergence follows from standard perturbation theory of integral equations (Muskhelishvili 1953). Equation (65) is equivalent to Eq. (41) upon differentiating twice.

Next we examine the limit $\ell \to 0$ with $\ell' = 0$. A similar limit has been studied in Zhang *et al.* (1998) for a semi-infinite crack by using the Wiener-Hopf technique.

Let $\phi_c(x)$ be the solution of the Cauchy integral equation (64) and let $\triangle(x) = \phi(x) - \phi_c(x)$ be the difference. Then, by Eq. (41), $\triangle(x)$ satisfies the equation

$$\mathsf{H}[\triangle](x) = \ell^2 \mathsf{H}[\phi]''(x) = \mathsf{H}[\phi](x) - p(x)/G, \quad |x|<1.$$

Since $\triangle(x)$ integrates to zero on $[-1,1]$, we have the formula for $\triangle(x)$ (Peters 1963):

$$\triangle(x) = \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{x-t}(\mathsf{H}[\phi](t) - p(t)/G)dt. \tag{66}$$

Now we only need to show $\mathsf{H}[\phi] - p/G$ vanishes as $\ell \to 0$. For clarity and simplicity of the presentation, we consider the uniform loading $p(x) = p_0$ and $p_0/G = 1$ for which the crack profile of the classical elasticity is the unit semi-circle. In this case, Eq. (42) becomes

$$\mathsf{H}[\phi](x) = 1 - \mathrm{e}^{1/\ell}\mathrm{e}^{x/\ell}/2 - \mathrm{e}^{-1/\ell}\mathrm{e}^{-x/\ell}/2 + C_1\mathrm{e}^{x/\ell} + C_2\mathrm{e}^{-x/\ell} = 1 + C_1'\mathrm{e}^{x/\ell} + C_2'\mathrm{e}^{-x/\ell}$$

where $C_1 = C_1' + \mathrm{e}^{1/\ell}/2, C_2 = C_2' + \mathrm{e}^{-1/\ell}/2$ satisfy

$$\begin{aligned}
\frac{C_1'}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\mathrm{e}^{t/\ell}dt + \frac{C_2'}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\mathrm{e}^{-t/\ell}dt + \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}dt &= 0\\
\frac{C_1'}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\mathrm{e}^{t/\ell}dt + \frac{C_2'}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\mathrm{e}^{-t/\ell}dt + \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}dt &= 0
\end{aligned}$$

following from Eq. (49) and (50). It is easy to see

$$C_1' = C_2' = -\left[\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\left(\mathrm{e}^{t/\ell} + \mathrm{e}^{-t/\ell}\right)dt\right]^{-1}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}dt. \tag{67}$$

Thus, the right side of Eq. (66) becomes

$$\frac{C_1'}{\pi\sqrt{1-x^2}}\left\{\int_{-1}^{1}\frac{\sqrt{1-t^2}\mathrm{e}^{t/\ell}}{x-t}dt+\int_{-1}^{1}\frac{\sqrt{1-t^2}\mathrm{e}^{-t/\ell}}{x-t}dt\right\}. \tag{68}$$

Clearly, as $\ell\to 0$, the dominant contribution to the first integral in (68) comes from $t\sim 1$ whereas the dominant contribution to the second integral comes from $t\sim -1$. The asymptotics of these integrals, as $\ell\to 0$, are straightforward:

$$\begin{aligned}\int_{-1}^{1}\frac{\sqrt{1-t^2}\mathrm{e}^{t/\ell}}{x-t}dt &\sim \sqrt{2}\mathrm{e}^{1/\ell}\ell^{3/2}\int_0^\infty\frac{\mathrm{e}^{-s}\sqrt{s}}{x-1+\ell s}ds\\ \int_{-1}^{1}\frac{\sqrt{1-t^2}\mathrm{e}^{-t/\ell}}{x-t}dt &\sim \sqrt{2}\mathrm{e}^{1/\ell}\ell^{3/2}\int_0^\infty\frac{\mathrm{e}^{-s}\sqrt{s}}{x+1-\ell s}ds.\end{aligned}$$

So Eq. (68) is asymptotically equivalent to

$$C_1'\frac{2\sqrt{2}\mathrm{e}^{1/\ell}\ell^{3/2}x}{\pi\sqrt{1-x^2}}\int_0^\infty\frac{\mathrm{e}^{-s}\sqrt{s}}{x^2-(1-\ell s)^2}ds=\begin{cases}O(\mathrm{e}^{1/\ell}\ell^{3/2}/\sqrt{1-x^2}), & \text{for } 1-x^2\geq n\ell\log 1/\ell\\ O(\mathrm{e}^{1/\ell}\ell^{1/2}/\sqrt{1-x^2}), & \text{otherwise}\end{cases} \tag{69}$$

for some sufficiently large $n>0$ (independent of ℓ). On the other hand, the relevant integrals in (67) have the following asymptotics

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\mathrm{e}^{t/\ell}dt\sim\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\mathrm{e}^{-t/\ell}dt\sim\sqrt{2}\sqrt{\ell}\mathrm{e}^{1/\ell}\int_0^\infty\frac{\mathrm{e}^{-s}}{\sqrt{s}}ds. \tag{70}$$

Put together, Eq. (69), (70) and (66) imply the following bound on $\triangle(x)$:

$$\triangle(x)=\begin{cases}O(\ell(1-x^2)^{-1/2}), & 1-x^2\geq n\ell\log 1/\ell\\ O((1-x^2)^{-1/2}), & \text{otherwise}\end{cases} \tag{71}$$

uniformly for some sufficiently large $n>0$ (independent of ℓ). The first estimate of (71) provides a *rate* of convergence of crack profile to the classical case away from the crack-tips. The second estimate of (71), in conjunction with the following other estimate, gives control over the behaviors in the immediate vicinity of the crack-tips.

To analyze the limiting behaviors in the neighborhood $1-x^2\leq n\ell\log 1/\ell$ of the crack-tips we use the alternative form (47) of solution. For $p/G=1$, we have $f(x)=-1+e^{1/\ell}e^{x/\ell}/2+e^{-1/\ell}e^{-x/\ell}/2$ and

$$\phi(x)=\frac{\sqrt{1-x^2}}{\pi}\left\{\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}\left(2\int_{-1}^{1}\frac{\mathrm{e}^{t/\ell}}{\sqrt{1-t^2}}dt\right)^{-1}\int_{-1}^{1}\frac{\mathrm{e}^{t/\ell}+\mathrm{e}^{-t/\ell}}{\sqrt{1-t^2}(x-t)}dt-\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}(x-t)}\right\}.$$

A similar asymptotic analysis gives the following estimate on the slope $\phi(x)$

$$\phi(x)\sim\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}\left(\int_0^\infty\frac{\mathrm{e}^{-s}}{\sqrt{s}}ds\right)^{-1}\frac{x\sqrt{1-x^2}}{\pi}\int_0^\infty\frac{\mathrm{e}^{-s}}{\sqrt{s}(x^2-(1-\ell s)^2)}ds.$$

In contrast to (69), the relevant asymptotics is now

$$\sqrt{1-x^2}\int_0^\infty\frac{\mathrm{e}^{-s}}{\sqrt{s}(x^2-(1-\ell s)^2)}ds=\begin{cases}O(\sqrt{1-x^2}), & \text{for } 1-x^2\geq n\ell\log 1/\ell\\ O(\ell^{-3/2}\sqrt{1-x^2}), & \text{otherwise}\end{cases} \tag{72}$$

for some sufficiently large n (independent of ℓ). Note that the second estimate of (71) holds for $\phi(x)$ as it does for $\phi_c(x)$.

Now we can close the proof by applying the mean value theorem to the crack displacement near the crack-tips: $w(x) = w(-1) + \phi(\bar{x}_1)(1 + x)$, $w(x) = w(1) + \phi(\bar{x}_2)(x - 1)$ for some $\bar{x}_1 \in (-1, x), \bar{x}_2 \in (x, 1)$ by choosing x. The second estimates of (71) and (72) together imply that the displacement in the region $|x^2 - 1| \leq n\ell \log 1/\ell$ is uniformly bounded by

$$C \min\left\{\ell^{-3/2}(1 - x^2)^{3/2}, (1 - x^2)^{1/2}\right\}, \quad \text{for } |x^2 - 1| \leq n\ell \log 1/\ell,$$

which vanishes, as $\ell \to 0$, uniformly in the corresponding region as did the classical crack *displacement.*

9 Numerical results

Our numerical solutions of Eq. (37) employ the fast Fourier transform and the collocation method in terms of the Chebyshev polynomials. The results are shown in Figure 1, 2, 4, 5, 6, 7 and Table 1.

- Figure 1 and 2 show that for $\rho > -1$ no oscillations occur in the crack profile. We see that the crack profiles of the gradient elasticity have *cusps* at the crack-tips and are consistent with the analytical results of Section 6.
- Figure 2 also shows that for $\rho < -1$ oscillations as well as negative displacements (i.e. displacement opposite to the loading condition) arise and, for $\rho = -1$, the profile is not stable.
- Figure 4 shows the convergence of the slopes to the classical counterpart, as $\ell \to 0$, in the region away from the crack-tips, for $\rho = 0.5$. At the crack-tips, the slopes are zero in contrast to the infinite slopes of the classical profile. Similar convergence holds for other values of $\rho > -1$. This is consistent with the analytical results of Section 8.
- Figure 5 shows the convergence of the stresses to the classical counterpart, as $\ell \to 0$, in the region away from the crack-tips, for $\rho = 0.5$. We see that, as the crack-tip is approached, the stresses change sign and become negative. Near the crack-tips, the stresses of the gradient elasticity are more singular than their classical counterpart. Similar convergence holds for other values of $\rho > -1$. This is consistent with the analytical results of Section 8.
- Figure 6 shows, for $\rho < -1$, the slopes of the crack do *not* converge. Instead, oscillations develop and, as $\ell \to 0$, become more severe. Similar divergence occurs for other values of $\rho < -1$.
- Table 1 indicates the convergence of the SIFs to the *negative* of the classical counterpart for $\rho = 0.1, 1.5$ and the *divergence* of the SIFs for $\rho = -1.5$, as $\ell \to 0$.

10 Conclusions

We have considered Casal's strain gradient elasticity with two material lengths ℓ, ℓ' for a Mode III finite crack which gives rise to a higher-order elliptic mixed boundary value problem. We take the boundary integral formulation and transform the problem into a hypersingular integrodifferential equation on the crack line supplemented with the natural crack-tip conditions.

For a particular type of boundary condition we have obtained explicitly the Green function. The full field solution is then expressed in terms of the crack profile and the Green function. For $\ell' = 0$,

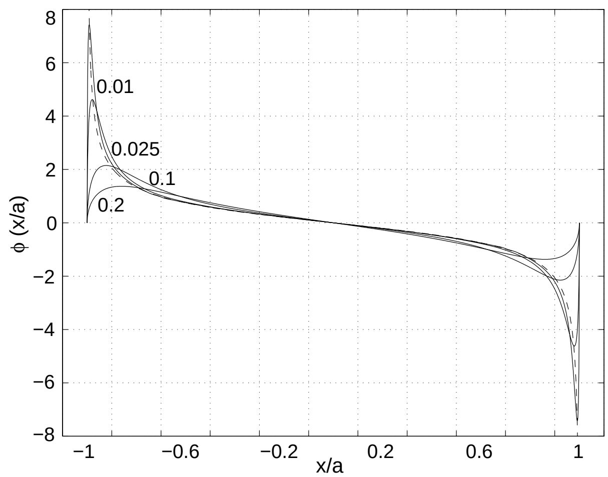

Figure 4: Slope of the crack for fixed $\rho = 0.5$ and various ℓ as indicated. The dashed curve is the slope for the classical elasticity. Similar convergence holds for other values of $\rho > -1$.

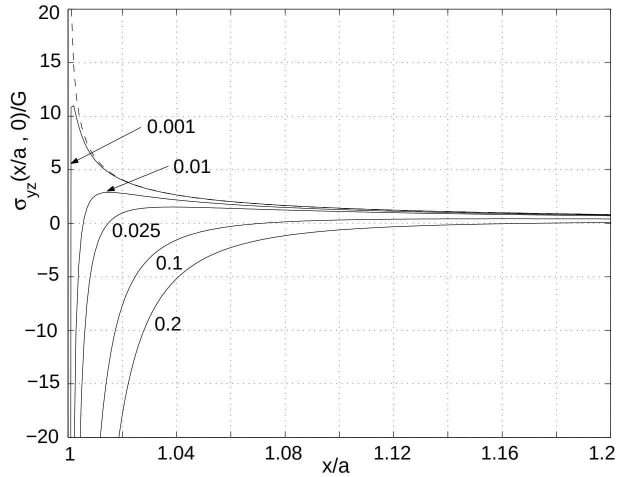

Figure 5: Stress $\sigma_{yz}(x/a, 0)/G$ along the ligament for fixed $\rho = 0.5$ and various ℓ as indicated The dashed curve is the stress for the classical elasticity. Similar convergence holds for other values of $\rho > -1$.

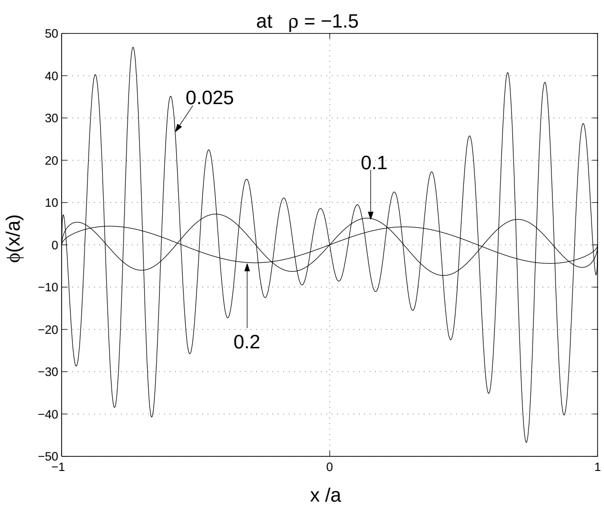

Figure 6: The slope of the crack profiles for fixed $\rho = -1.5$ and various ℓ as indicated.

ρ	$\ell = 0.1$	$\ell = 0.05$	$\ell = 0.025$
0.1	-0.9438012	-0.9720323	-0.9863192
1.5	-0.9155931	-0.9647259	-1.0015685
-1.5	-1.4612889	-0.3336649	-0.4651179

Table 1: Normalized SIFs $\frac{K_{III}(a)}{p_0\sqrt{2\pi a}}$.

we have obtained a closed-form solution for the crack profile in two alternative forms which yield explicitly the crack-tip asymptotics $O((1-x^2/a^2)^{3/2})$ for the displacement and $O((1-x^2/a^2)^{-3/2})$ for the stress. The case of small ℓ' is shown to be a regular perturbation of the case $\ell'=0$ and, thus, shares the same type of asymptotics. Numerical solutions are given for various values of $\rho \neq -1$.

For the limit $\ell \to 0$ with $\rho \neq -1$ fixed we show the convergence of the Green function to its classical counterpart. When $\rho = -1$, the Green function does *not* converge to the classical one. For small ρ, we have shown analytically the full field convergence of displacement to its the classical counterpart. For arbitrary $\rho > -1$ we show numerically the convergence of the crack profile, the slope and the stress. For $\rho < -1$, numerical evidences point to the *divergence* of the crack profiles and the slopes.

Moreover, the numerical calculation indicates the convergence of a suitably defined SIF to the negative of the classical counterpart for $\rho > -1$, as $\ell \to 0$, even though the stresses are one-order more singular than the classical stresses near the crack-tips. The SIFs for $\rho < -1$ are shown to oscillate and diverge as $\ell \to 0$.

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Appendices

A Finite-part integrals

Let $C^\alpha(-1,1)$ be the space of *locally* Hölder continuous functions on $(-1,1)$ with index $\alpha \le 1$. Denote $L^{1+} = \bigcup_{p>1} L^p[-1,1]$.

Definition 1 (Cauchy principal value integral)

$$\fint_{-1}^{1} \frac{\phi(t)}{t-x} dt := \lim_{\epsilon \to 0} \left\{ \int_{-1}^{x-\epsilon} \frac{\phi(t)}{t-x} dt + \int_{x+\epsilon}^{1} \frac{\phi(t)}{t-x} dt \right\}, \quad |x| < 1,$$

for any $\phi \in C^\alpha(-1,1) \cap L^{1+}, 0 < \alpha \le 1$.

By definition, we have

$$\begin{aligned} \fint_{-1}^{1} \frac{\phi(t)}{t-x} dt &= \lim_{\epsilon \to 0} \left\{ \int_{|t-x|\ge\epsilon} \frac{\phi(t)-\phi(x)}{t-x} dt + \phi(x) \int_{|t-x|\ge\epsilon} \frac{dt}{t-x} \right\} \\ &= \int_{-1}^{1} \frac{\phi(t)-\phi(x)}{t-x} dt + \phi(x) \fint_{-1}^{1} \frac{dt}{t-x}. \end{aligned} \tag{73}$$

Note that for any $\phi \in C^\alpha, \alpha > 0$, the first integral on the right side of (73) is an ordinary Riemann integral and the second integral is

$$\fint_{-1}^{1} \frac{dt}{t-x} = \log \frac{1-x}{1+x}, \quad |x| < 1.$$

Denote by $C^{m,\alpha}(-1,1)$ the space of functions whose m-th derivatives are *locally* Hölder continuous with index $0 < \alpha \le 1$. Finite-part integrals are defined recursively as follows.

Definition 2 (Finite-part integral) *For any* $\phi \in C^{n,\alpha}(-1,1) \cap L^{1+}$ *and* $n = 1, 2, 3, ...$

$$\int\!\!\!\!\!\!=_{-1}^{1} \frac{\phi(t)}{(t-x)^{n+1}}dt := \frac{1}{n}\frac{d}{dx} \int\!\!\!\!\!\!=_{-1}^{1} \frac{\phi(t)}{(t-x)^{n}}dt, \quad |x| < 1,$$

with

$$\int\!\!\!\!\!\!=_{-1}^{1} \frac{\phi(t)}{t-x}dt := \int\!\!\!\!\!\!-_{-1}^{1} \frac{\phi(t)}{t-x}dt.$$

¿From (73) and the definition of finite-part integrals, it follows that

$$\begin{aligned} & \int\!\!\!\!\!\!=_{-1}^{1} \frac{\phi(t)}{(t-x)^{n}}dt \\ = & \int_{-1}^{1} \frac{\phi(t) - \sum_{j=0}^{n-1} \phi^{(j)}(x)(t-x)^j/k!}{(t-x)^n} + \sum_{j=0}^{n-1} \phi^{(j)}(x)/k! \int\!\!\!\!\!\!=_{-1}^{1} \frac{dt}{(t-x)^{n-j}}. \end{aligned} \tag{74}$$

For $\phi \in C^{n,\alpha}(-1,1) \cap L^{1+}$, the first integral on the right side of (74) is an ordinary Riemann integral. It is easy to check that the integration-by-parts formula holds for finite part integrals.

Proposition 1 *For* $\phi \in C^{n,\alpha}(-1,1) \cap L^{1+}$

$$\int\!\!\!\!\!\!=_{-1}^{1} \frac{\phi'(t)}{(t-x)^{n}}dt = n \int\!\!\!\!\!\!=_{-1}^{1} \frac{\phi(t)}{(t-x)^{n+1}}dt + \frac{\phi(1)}{(1-x)^n} - (-1)^n \frac{\phi(-1)}{(1+x)^n}, \quad n \geq 1$$

and for $\phi \in C^{\alpha}(-1,1) \cap L^{1+}$

$$\int_{-1}^{1} \phi'(t) \log|t-x| dt = \int\!\!\!\!\!\!-_{-1}^{1} \frac{\phi(t)}{t-x}dt + \phi(1)\log|1-x| - \phi(-1)\log|1+x|$$

Alternatively, one can define finite-part integrals by equation (74) and deduce Definition 2 and Proposition 1 as properties.

B Hypersingular kernels

For the derivation of hypersingular kernels, we use three basic ingredients:

- definition of finite part integrals;
- the following identity

$$\frac{d^n}{dy^n}\left(\frac{1}{y - i(t-x)}\right) = i^{-n}\frac{d^n}{dx^n}\left(\frac{1}{y - i(t-x)}\right); \tag{75}$$

- the Plemelj formula (Elliott 1951; Muskhelishvili 1953).

Proposition 2

$$\lim_{\epsilon \to 0} \int_{-1}^{1} \frac{\phi(t)}{(t-x) + i\epsilon}dt = \int\!\!\!\!\!\!-_{-1}^{1} \frac{\phi(t)}{t-x}dt + \pi i \phi(x), \quad \phi \in L^{1+}.$$

Observe that

$$\begin{aligned} k_n(t-x,y) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} i^n|\xi|^n \frac{|\xi|}{i\xi} \mathrm{e}^{-|\xi|y+i(t-x)\xi}\ d\xi \\ &= \sqrt{\frac{2}{\pi}}(-i)^n Im\left[\frac{d^n}{dy^n}(y-i(t-x))^{-1}\right] \\ &= (-1)^n\sqrt{\frac{2}{\pi}} Im\left[\frac{d^n}{dx^n}(y-i(t-x))^{-1}\right] \\ &= (-1)^n\sqrt{\frac{2}{\pi}} Re\left[\frac{d^n}{dx^n}(t-x+iy)^{-1}\right]. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{y\to 0^+}\int_{-1}^{1} k_n(t-x,y)\phi(t)\ dt &= \lim_{y\to 0^+}(-1)^n\sqrt{\frac{2}{\pi}}\int_{-1}^{1} Re\left[\frac{d^n}{dx^n}(t-x+iy)^{-1}\right]\phi(t)\ dt \\ &= (-1)^n\sqrt{\frac{2}{\pi}} Re\left[\frac{d^n}{dx^n}\lim_{y\to 0^+}\int_{-1}^{1}(t-x+iy)^{-1}\phi(t)\ dt\right] \\ &= (-1)^n\sqrt{\frac{2}{\pi}}\frac{d^n}{dx^n}\int_{-1}^{1}\frac{\phi(t)}{t-x}\ dt \\ &= n!(-1)^n\sqrt{\frac{2}{\pi}}\;⨎_{-1}^{1}\frac{\phi(t)}{(t-x)^{n+1}}\ dt. \end{aligned}$$

by the Plemelj formula and the definition of finite part integrals.
Note that, when n is an odd integer,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i^n\xi^n\frac{|\xi|}{i\xi}e^{-|\xi|y+i(t-x)\xi}\ d\xi = -\sqrt{\frac{2}{\pi}} Im\left[\frac{d^n}{dx^n}(t-x+iy)^{-1}\right].$$

Thus we have

$$\begin{aligned} &\int_{-1}^{1} dt\phi(t)\lim_{y\to 0^+}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i^n\xi^n\frac{|\xi|}{i\xi}e^{-|\xi|y+i(t-x)\xi}\ d\xi \\ &= -\sqrt{\frac{2}{\pi}} Im\left[\frac{d^n}{dx^n}\lim_{y\to 0^+}\int_{-1}^{1}\phi(t)(t-x+iy)^{-1}\ dt\right] \\ &= -\sqrt{2\pi}\frac{d^n}{dx^n}\phi(x) \end{aligned}$$

where we have used the Plemelj formula again.

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