COMPRESSIVE IMAGING OF SUBWAVELENGTH STRUCTURES

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Abstract. The problem of imaging periodic structure (source or scatterer) is formulated in the framework of compressed sensing with special attention on subwavelength resolution. It is shown that in this formulation the subwavelength resolution in the presence of noise can not be achieved by compressed sensing techniques alone. Additional techniques such as near-field measurement or illumination are required and the resolution limit is derived, which says that the smallest stably resolvable scale is about either half the wavelength or the distance of the sensors to the target, whichever is smaller. Numerical simulations are consistent with the theoretical prediction.

1. Introduction

The Compressed Sensing paradigm has supplied a fresh perspective on the imaging problems, including source inversion and inverse scattering (see the extensive literature in [6]).

This brief note is to (i) formulate the imaging problem of inverse source and scattering of periodic structures in the framework of compressed sensing and analyze it by the fundamental results of Candès, Romberg and Tao [2, 3, 4, 5] and Rudelson and Vershynin [13] on Fourier measurement; (ii) use the solution thus obtained as a premise to investigate the problem of subwavelength resolution in the presence of noise. In particular, we discuss the idea of extracting subwavelength information by near-field measurement and illumination, common in nano-optics [12].

Finally we present numerical results confirming the theoretical predictions and conclude with a few remarks.

2. Source inversion

First we consider imaging of an extended source (the target) in two dimensional (x, z)-plane. We assume that the target is located in the line $z = z_0$.

The target is represented by the variable source amplitude $\sigma(x)$ which is assumed to be periodic with period L and admits the Fourier expansion

$$\sigma(x) = \sum_{k=-\infty}^{\infty} \hat{\sigma}_k e^{i2\pi kx/L}. \tag{1}$$

We consider the class of band-limited functions with $\hat{\sigma}_k = 0$ except for

$$k \in [-(m-1)/2, (m-1)/2] \tag{2}$$

in (1) where m is an odd integer so that there are at most m relevant Fourier modes.

The wave propagation in the free space is governed by the Helmholtz equation

$$\Delta u + \omega^2 u = 0.$$

where $\omega = 2\pi/\lambda$ is the frequency assuming the wave speed is unity. In two dimensions, the Green function $G(\mathbf{r})$ is

$$G(\mathbf{r}) = \frac{-i}{4} H_0^{(1)}(\omega|\mathbf{r}|), \quad \mathbf{r} = (x, z)$$

where $H_0^{(1)}$ is the zeroth order Hankel function of the first kind. G can be expressed by the Sommerfeld integral representation

$$G(\mathbf{r}) = \frac{-i}{4\pi} \int e^{i\omega(|z|\beta(\alpha)+x\alpha)} \frac{d\alpha}{\beta(\alpha)}, \tag{3}$$

where

$$\beta(\alpha) = \begin{cases} \sqrt{1-\alpha^2}, & |\alpha| < 1 \\ i\sqrt{\alpha^2 - 1}, & |\alpha| > 1 \end{cases} \tag{4}$$

[1]. The integrand in (3) with real-valued β (i.e. $|\alpha| < 1$) corresponds to the homogeneous wave and that with imaginary-valued β (i.e. $|\alpha| > 1$) corresponds to the evanescent (inhomogeneous) wave which has an exponential-decay factor $e^{-\omega|z|\sqrt{\alpha-1}}$.

The present framework can be easily extended to three dimensions by using the plane-wave representation

$$\frac{e^{i\omega|\mathbf{r}|}}{|\mathbf{r}|} = \frac{i\omega}{2\pi} \int \frac{d\alpha d\beta}{\gamma} \exp\left[i\omega(\alpha x + \beta y + \gamma|z|)\right], \quad \mathbf{r} = (x, y, z) \tag{5}$$

where

$$\begin{aligned} \gamma &= \sqrt{1-\alpha^2-\beta^2}, \quad \alpha^2+\beta^2 \le 1 \\ \gamma &= i\sqrt{\alpha^2+\beta^2-1}, \quad \alpha^2+\beta^2 > 1 \end{aligned}$$

for the Green function [1]. For simplicity of the presentation, we stick with the two dimensional setting throughout.

The signal arriving at the sensor located at $(0, \xi)$ is given by

$$\int G(z_0, x-\xi)\sigma(x)dx = -\frac{i}{2\omega} \sum_k \frac{\hat{\sigma}_k}{\beta_k} e^{i\omega z_0 \beta_k} e^{i\omega\alpha_k\xi} \tag{6}$$

by (3) where

$$\alpha_k = \frac{k\lambda}{L}, \qquad \beta_k = \beta(\alpha_k). \tag{7}$$

By (7), the subwavelength structure of the target is encoded in $\hat{\sigma}_k$ with $|k|\lambda > L$ and carried by the evanescent wave.

Let $\mathbf{a}_i = (0, \xi_i), i = 1, ..., n$ be the coordinates of the n sensors in the line $z = 0$. To set the problem in the framework of compressed sensing we define the n received signals by the sensors as the measurement vector Y and set the target vector $X = (X_k) \in \mathbb{C}^m$ as

$$X_k = -\frac{i\sqrt{n}e^{i\omega z_0 \beta_k}}{2\omega\beta_k}\hat{\sigma}_k. \tag{8}$$

This leads to the form

$$Y = \mathbf{A}X \tag{9}$$

where the sensing matrix $\mathbf{A} = [A_{jk}] \in \mathbb{C}^{n\times m}$ has unit columns with entries

$$A_{jk} \quad = \quad \frac{1}{\sqrt{n}}e^{i2\pi k\xi_j/L}, \quad j = 1, ..., n, \quad k = -\frac{m-1}{2}, ..., \frac{m-1}{2}. \tag{10}$$

Note that the formulation is exact and no approximation is made. To avoid a vanishing denominator in (8), we must set $L/\lambda \notin \mathbb{N}$ so that $\beta_k \neq 0$.

The main thrust of compressive sensing is that under suitable conditions the inversion can be achieved as the ℓ^1-minimization

$$\min\|X\|_1, \quad \text{subject to} \quad Y = \mathbf{A}X$$

which also goes by the name of *Basis Pursuit* (BP) and can be solved by linear programming. BP was first introduced empirically in seismology by Claerbout and Muir and later studied mathematically by Donoho and others [7, 9].

A fundamental notion in compressed sensing under which BP yields the unique exact solution is the restrictive isometry property due to Candès and Tao [4]. Precisely, let the sparsity s of the target vector be the number of nonzero components of X and define the restricted isometry constant δ_s to be the smallest positive number such that the inequality

$$(1-\delta_s)\|Z\|_2^2 \le \|\mathbf{A}Z\|_2^2 \le (1+\delta_s)\|Z\|_2^2$$

holds for all $Z \in \mathbb{C}^m$ of sparsity at most s.

To realize the restrictive isometry property for our matrix $\mathbf{A}$, we follow [2] and let $I_1, I_2, ..., I_m$ be independent Bernoulli random variables taking the value 1 with probability $\bar{n}/m, \bar{n} \ll m$. Let

$$\xi_j = \eta_j L/m \tag{11}$$

where η_j belongs to the random set

$$\Omega = \{q \in \{0, ..., m-1\} | I_q = 1\}. \tag{12}$$

We have $\mathbb{E}|\Omega| = \bar{n}$ and indeed by Bernstein's inequality $n = |\Omega|$ is close to $\bar{n}$ with high probability.

Now we state two fundamental results essential for our problem.

Theorem 1. [4] *Suppose the true target vector X has the sparsity at most s. Suppose the restricted isometry constant of $\mathbf{A}$ satisfies the inequality*

$$\delta_{3s} + 3\delta_{4s} < 2. \tag{13}$$

Then X is the unique solution of BP.

Remark 1. *Greedy algorithms have significantly lower computational complexity than linear programming and have provable performance under various conditions. For example under the condition $\delta_{3s} < 0.06$ the Subspace Pursuit (SP) algorithm is guaranteed to exactly recover X via a finite number of iterations* [10].

Theorem 2. [13] *For any $\tau > 1$ and any $m, s > 2$, a random set Ω of average cardinality*

$$\bar{n} = (C\tau s \log m) \log (C\tau s \log m) \log^2 s \tag{14}$$

gives rise to a sensing matrix $\mathbf{A}$ which satisfies the restricted isometry condition (13) with probability at least $1 - 5e^{-c\tau}$. Here c and C are some absolute constants.

As a consequence of Theorem 1 and 2, the target vector X can be determined by BP and the source amplitude $\sigma(x)$ can be reconstructed exactly from (8), including all the subwavelength structures corresponding to $|\alpha_k| > 1$. In other words, in the absence of noise there is essentially no limitation to the resolving power of the compressed sensing technique, subwavelength or not, as long as sufficient number of measurements are made.

When noise is present, however, the performance of the above approach may be severely limited, especially in the recoverability of subwavelength information. Consider the standard model of additive noise

$$Y^\varepsilon = \mathbf{A}X + E \tag{15}$$

where $\|E\|_2 = \varepsilon > 0$ is the size of the noise and the associated relaxation scheme

$$\min \|X\|_1, \quad \text{subject to} \quad \|Y^\varepsilon - \mathbf{A}X\|_2 \leq \epsilon. \tag{16}$$

Theorem 3. [3] *Let X^ε be the solution to (16). If $\|X\|_0 \leq s$ and (13) is true, then*

$$\|X^\varepsilon - X\|_2 \leq C_s \varepsilon$$

where C_s depends only on and behaves reasonably with δ_{4s}.

See also [8, 14] for stability result under the condition of incoherence.

Inverting the relationship (8) with small error in the target vector X produces a controllable error for those $\hat{\sigma}_k$ such that

$$|e^{i\omega z_0 \beta_k}| \geq e^{-2\pi} \tag{17}$$

but exponentially amplified error otherwise. The stably resolvable scales include those corresponding to $|\alpha_k| \leq 1$ as well as $|\alpha_k| > 1$ such that

$$|\beta_k| z_0 \leq \lambda. \tag{18}$$

A simple scheme to regularize the reconstruction is to set the unstable modes to be zero. The choice of the stability threshold $e^{2\pi}$ for the noise amplification factor is convenient but arbitrary; any constant less than one will lead to similar conclusions.

Condition (18) leads to the inequality

$$\alpha_k = \frac{k\lambda}{L} \leq \sqrt{1 + \left(\frac{\lambda}{z_0}\right)^2} \tag{19}$$

and hence the smallest stably resolvable scale for the target profile σ is

$$\left(\frac{1}{\lambda^2}+\frac{1}{z_0^2}\right)^{-1/2}. \tag{20}$$

One way of overcoming this difficulty is to reduce the distance z_0 between the sensor array and the target. When $z_0 < \lambda$ then the above discussion implies that the smallest stably resolvable scale is about z_0. This is the near-field imaging and is the idea behind the scanning microscopy.

Next let us discuss a different mechanism of superresolution available in the context of inverse scattering.

3. Inverse scattering

Consider a periodic scatterer with scattering amplitude σ admitting the representation (1). For simplicity, we use the Born scattering model [1] under which the scattered field at $z=0$ is given by

$$u_s(0,\xi)=\int G(z_0,x-\xi)\sigma(x)u_i(z_0,x)dx \tag{21}$$

where $u_i(z,x)$ is the incident field. Now with the normally incident plane wave $u_i(z,x)=e^{i\omega z}$, eq. (21) is essentially reduced to (6).

Consider the obliquely incident plane wave $u_i(z,x)=e^{i\omega(\alpha x+\beta|z-z_1|)}$ where β is related to α as in (7) and z_1, assumed larger than z_0. Set

$$\alpha=\frac{q\lambda}{L},\quad q\in\mathbb{R}.$$

In the standard setting the illumination field is a homogeneous wave with $|\alpha|<1$. The evanescent illumination $|\alpha|>1$ will also be considered here. In such case, z_1 is the z-coordinate of the illumination source.

The same calculation as before now leads to

$$u_s(0,\xi)=-\frac{i}{2\omega\sqrt{m}}\sum_k\frac{\hat{\sigma}_k}{\beta_k}e^{i\omega z_0\beta_k}e^{i\omega(z_1-z_0)\beta}e^{i\omega\alpha_k\xi}$$

where instead of (7) we have

$$\alpha_k=\frac{(k+q)\lambda}{L},\qquad \beta_k=\beta(\alpha_k).$$

Set the target vector $X^{(q)}=(X_k^{(q)})\in\mathbb{C}^m$ as

$$X_k^{(q)}=-\frac{ie^{i\omega z_0\beta_k}e^{i\omega(z_1-z_0)\beta}}{2\omega\beta_k}\hat{\sigma}_k \tag{22}$$

and proceed as before. To avoid a vanishing denominator we require

$$\frac{(k+q)\lambda}{L}\neq 1,\quad \forall k. \tag{23}$$

Theorem 1 and 2 are applicable to the shifted Fourier matrix $\mathbf{A}^{(q)}$ with entries

$$A^{(q)}_{jk} = \frac{1}{\sqrt{m}} e^{i\omega(k+q)\lambda\xi_j/L}$$

where ξ_j is given by (11) and the target vector $X^{(q)}$ can be recovered exactly in the absence of noise.

The discussion of Section 2 implies that for $\hat{\sigma}_k$ to be resolvable in Born scattering inversion in the presence of small noise

$$|\alpha_k| = \frac{|k+q|\lambda}{L} \leq \sqrt{1+\frac{\lambda^2}{z_0^2}}. \tag{24}$$

For each q, the stably resolvable modes satisfy

$$-q-\sqrt{\frac{L^2}{\lambda^2}+\frac{L^2}{z_0^2}} \leq k \leq -q+\sqrt{\frac{L^2}{\lambda^2}+\frac{L^2}{z_0^2}} \tag{25}$$

and the smallest stably resolvable scale is about

$$\left(\frac{|q|}{L}+\sqrt{\frac{1}{\lambda^2}+\frac{1}{z_0^2}}\right)^{-1}. \tag{26}$$

Minimizing (26) under the constraint $|e^{i\omega(z_1-z_0)\beta}| > e^{-2\pi}$ then yields the resolution limit of the method.

Consider two different illuminations: homogeneous and evanescent wave sources. If the target is illuminated by an incident homogeneous wave, then $|q|/L < \lambda^{-1}$ and the smallest resolvable scale is about

$$\left(\frac{1}{\lambda}+\sqrt{\frac{1}{\lambda^2}+\frac{1}{z_0^2}}\right)^{-1}. \tag{27}$$

In far-field measurement ($z_0 \gg \lambda$), the resolution limit (27) is about $\lambda/2$.

On the other hand, if the incident wave is evanescent and subject to the constraint $|e^{i\omega(z_1-z_0)\beta}| > e^{-2\pi}$, which implies

$$|q| < \sqrt{\frac{L^2}{\lambda^2}+\frac{L^2}{(z_1-z_0)^2}}, \tag{28}$$

then the smallest stably resolvable scale according to (26) is about

$$\left(\sqrt{\frac{1}{\lambda^2}+\frac{1}{(z_1-z_0)^2}}+\sqrt{\frac{1}{\lambda^2}+\frac{1}{z_0^2}}\right)^{-1} \tag{29}$$

which for near-field illumination ($|z_1-z_0| \ll \lambda$) but far-field measurement ($z_0 \gg \lambda$) becomes $\mathcal{O}(z_1-z_0)$.

To achieve the resolution limit set by (29) we pick $q_* > 0$ sufficiently close to the right hand side of (28) and satisfying (23) and illuminate the target by a few evanescent waves with

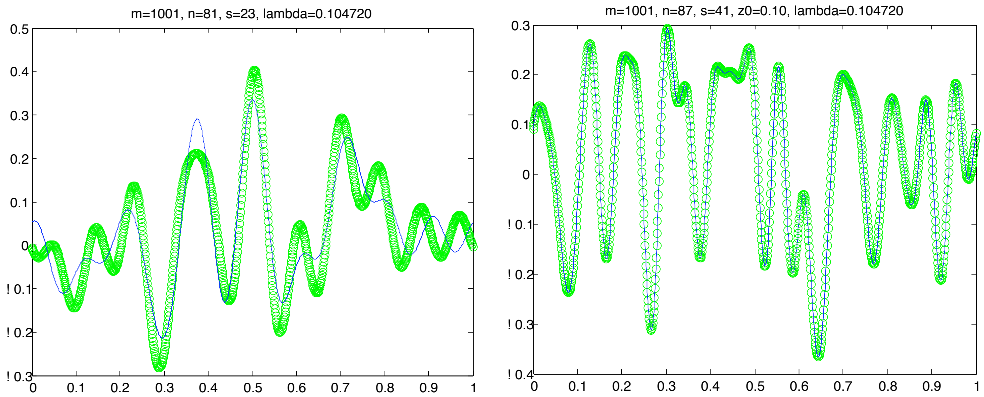

FIGURE 1. In source inversion with far-field measurement (left, $z_0 = 1, n = 81$), the two subwavelength modes $k = \pm 11$ cause significant errors. With near-field measurement (right, $z_0 = 0.1, n = 87$) the reconstruction is nearly perfect for target with 20 subwavelength modes. The green-circled curve is the original profile and the blue-solid curve is the reconstructed profile.

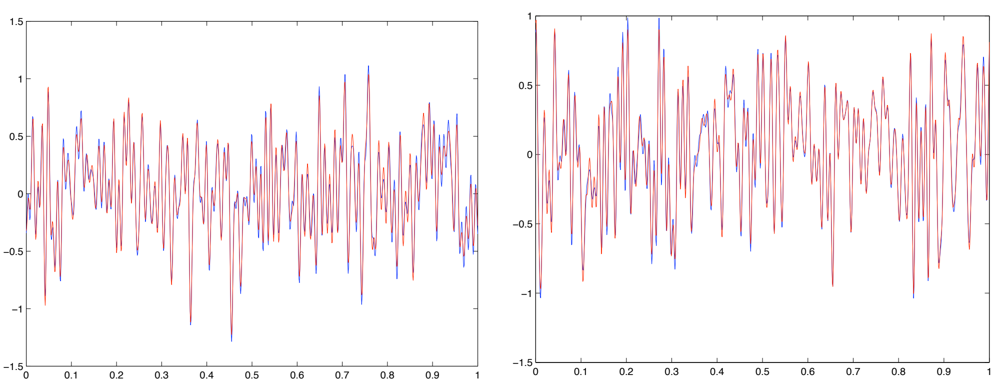

FIGURE 2. Two independent runs (left with $n = 70$ right with $n = 82$) with far-field measurement ($z_0 = 1$) and near-field illumination ($z_1 - z_0 = 0.01$). The red curve is the original profile and the blue curve is the reconstructed profile. Except for a few spots, they coincide with each other.

$q \in [-q_*, q_*]$. The union of their respective stably resolvable modes (25) is the total stably resolvable modes. Evanescent illumination can be produced physically by, for example, the total internal reflection in optics [12].

4. Numerical results

In our numerical simulations, $L = 1$, $\lambda = \pi/30$, $m = 1001$ so the subwavelength mode cutoff is at about $|k| = 10$. The roundoff error is the main source of noise in our simulations. We regularize the reconstruction by setting the unstable modes to zero.

Figure 1 shows the numerical results for source inversion using the Subspace Pursuit for the compressed sensing step. First we image a periodic source with 23 modes, including two subwavelength modes $k = \pm 11$ using far-field measurement $z_0 = 1$. We can see in the left plot of Figure 1 that the recovery of the subwavelength modes is not accurate. Reducing the distance between the target and the sensors ($z_0 = 0.1$) enables accurate reconstruction of a profile with 41 modes (right plot in Figure 1).

Figure 2 shows the results for inverse Born scattering by the far-field measurement ($z_0 = 1$) and the near-field illumination ($z_1 - z_0 = 0.01$). In this case it suffices to run the procedure for two incident modes with $q = \pm 90$. With similar number of sensors, the method can accurately recovers about 205 modes.

5. Conclusion

In this note, we have analyzed the problem of imaging periodic structures in the perspective of Compressed Sensing which provides assurance of stable reconstruction of a target composed of sparse Fourier modes with the similar number of measurements, modulo a poly-logarithmic factor, in the presence of noise.

However, Compressed Sensing alone does not achieve subwavelength resolution as the additional step required to recover the true target amplitude may amplify the noise exponentially.

We have discussed the imaging techniques of near-field measurement and illumination and derived the resolution limit, which says that the smallest stably resolvable scale is about either half the wavelength or the distance of the sensors to the target, whichever is smaller.

Finally we note that the compressive imaging theory in the remote sensing regime for discrete point targets has been recently developed in [11].

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References

[1] M. Born and E. Wolf, *Principles of Optics*, 7-th edition, Cambridge University Press, 1999.

[2] E. Candés, J. Romberg and T. Tao, "Robust undertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. Inform. Theory* **52** (2006), 489-509.

[3] E.J. Candès, J. Romberg and T. Tao, "Stable signal recovery from incomplete and inaccurate measurements," *Commun. Pure Appl. Math.* **59** (2006), 120723.

[4] E. J. Candès and T. Tao, " Decoding by linear programming," *IEEE Trans. Inform. Theory* **51** (2005), 42034215.

[5] E. J. Candès and T. Tao, " Near-optimal signal recovery from random projections: universal encoding strategies?," *IEEE Trans. Inform. Theory* **52** (2006), 54-6-5425.

[6] Compressive Sensing Resources. `http://www.dsp.ece.rice.edu/cs/`

[7] S.S. Chen, D.L. Donoho and M.A. Saunders, "Atomic decomposition by basis pursuit," *SIAM Rev.* **43** (2001), 129-159.

[8] D.L. Donoho, M. Elad and V.N. Temlyakov, "Stable recovery of sparse overcomplete representations in the presence of noise," *IEEE Trans. Inform. Theory* **52** (2006) 6-18.

[9] D.L. Donoho and X. Huo, "Uncertainty principle and ideal atomic decomposition, " *IEEE Trans. Inform. Theory* **47** (2001), 2845-2862.

[10] W. Dai and O. Milenkovic, "Subspace pursuit for compressive sensing: closing the gap between performance and complexity," arXiv:0803.0811.

[11] A.C. Fannjiang, P. Yan and T. Strohmer, "Compressed remote sensing of sparse objects," arXiv:0904.3994.

[12] L. Novotny and B. Hecht, *Principles of Nano-Optics*, Cambridge University Press, Cambridge, 2006.

[13] M. Rudelson and R. Vershynin, "On sparse reconstruction from Fourier and Gaussian measurements," *Comm. Pure Appl. Math.* **111** (2008) 1025-1045.

[14] J.A. Tropp, " Just relax: convex programming methods for identifying sparse signals in noise," *IEEE Trans. Inform. Theory* **52** (2006), 1030-1051. "Corrigendum" *IEEE Trans. Inform. Theory* (2008).

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