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Radiative transfer limit of two-frequency Wigner distribution for random parabolic waves: An exact solution [☆]

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Abstract

The present Note establishes the self-averaging, radiative transfer limit for the two-frequency Wigner distribution for classical waves in random media. Depending on the ratio of the wavelength to the correlation length the limiting equation is either a Boltzmann-like integral equation or a Fokker–Planck-like differential equation in the phase space. The limiting equation is used to estimate three physical parameters: the spatial spread, the coherence length and the coherence bandwidth. In the longitudinal case, the Fokker–Planck-like equation can be solved exactly. *To cite this article: A.C. Fannjiang, C. R. Physique 8 (2007).*

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Résumé

Limite du transfert radiatif de la distribution de Wigner à deux fréquences pour des ondes paraboliques aléatoires : une solution exacte. Dans cette Note nous établissons la limite auto-moyennante dans le régime du transfert radiatif pour la distribution de Wigner à deux fréquences dans le cas classique d'ondes en milieu aléatoires. Suivant le rapport de la longueur d'onde à la longueur de corrélation l'équation limite est soit une équation intégrale de type Boltzmann soit une équation différentielle de type Fokker–Planck dans l'espace des phases. L'équation limite est utilisée pour estimer trois paramètres physiques : l'étalement spatial, la longueur de cohérence et la largeur de bande cohérente. Dans le cas longitudinal l'équation de type Fokker–Planck admet une solution exacte. *Pour citer cet article : A.C. Fannjiang, C. R. Physique 8 (2007).*

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Keywords: Radiative transfer; Two-frequency Wigner distribution; Spatial spread; Coherence length; Coherence bandwidth

Mots-clés: Transfert radiatif; Distribution de Wigner à deux fréquences; Étalement spatial; Longueur de cohérence; Largeur de bande cohérente

1. Introduction

High-data-rate communication systems at millimeter and optical frequencies, remote sensing and detection and the astronomical imaging all require understanding of stochastic pulse propagation. As pulses consist of a typically broad

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frequency band the complete information about transient propagation requires a solution for the statistical moments of the wave field at different frequencies and locations [1].

Let k_1, k_2 be two (relative) wavenumbers nondimensionalized by the central wavenumber k_0 . Let the wave fields $\Psi_j, j = 1, 2$, of $k_j, j = 1, 2$, satisfy the parabolic (paraxial) wave equation in the dimensionless form [1]

$$i \frac{\partial}{\partial z} \Psi_j(z, \mathbf{x}) + \frac{\gamma}{2k_j} \nabla^2 \Psi_j(z, \mathbf{x}) + \frac{\mu k_j}{\gamma} V\left(\frac{z}{\delta}, \frac{\mathbf{x}}{\varepsilon}\right) \Psi_j(z, \mathbf{x}) = 0, \quad j = 1, 2 \tag{1}$$

where γ is the Fresnel number, V represents the refractive index fluctuation with the correlation lengths δ and ε in the longitudinal and transverse direction, respectively, and μ is the magnitude. Both δ and ε are small parameters related by the anisotropy parameter α as $\delta\alpha = \varepsilon$. When $\alpha \ll 1$ (resp. $\alpha \gg 1$), the refractive index fluctuates much faster (resp. slower) in the transverse direction(s) than in the longitudinal direction.

An important regime for classical wave propagation takes place when the correlation length is much smaller than the propagation distance but is comparable or much larger than the central wavelength which is proportional to the Fresnel number. This is the radiative transfer regime for monochromatic waves described by the following scaling limit

$$\gamma = \theta\varepsilon, \quad \mu = \sqrt{\varepsilon}, \quad \theta > 0, \quad \text{such that} \quad \lim_{\varepsilon \rightarrow 0} \theta < \infty \tag{2}$$

(see [2–5] and references therein). With two different frequencies, the most interesting scaling limit requires another simultaneous limit

$$\lim_{\varepsilon \rightarrow 0} k_1 = \lim_{\varepsilon \rightarrow 0} k_2 = k, \quad \lim_{\varepsilon \rightarrow 0} \gamma^{-1} k^{-1} (k_2 - k_1) = \beta > 0 \tag{3}$$

We shall refer to the conditions (2) and (3) as the two-frequency radiative transfer scaling limit.

The two-frequency mutual coherence function [1]

$$\Gamma_{12}(z, \mathbf{x}, \mathbf{y}) = \mathbb{E} \left[\Psi_1 \left(z, \mathbf{x} + \frac{\gamma \mathbf{y}}{2} \right) \Psi_2 \left(z, \mathbf{x} - \frac{\gamma \mathbf{y}}{2} \right) \right]$$

where \mathbb{E} stands for the ensemble averaging, plays an important role in analyzing propagation of random pulses [1]. But in the radiative transfer regime the two-frequency mutual coherence function is not as convenient as the two-frequency Wigner distribution, introduced in [6], which is a natural extension of the standard Wigner distribution and is self-averaging in the radiative transfer regime.

1.1. Two-frequency Wigner distribution

The two-frequency Wigner distribution is defined as

$$W_z(\mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^d} \int e^{-i\mathbf{p}\cdot\mathbf{y}} \Psi_1 \left(z, \frac{\mathbf{x}}{\sqrt{k_1}} + \frac{\gamma \mathbf{y}}{2\sqrt{k_1}} \right) \Psi_2^* \left(z, \frac{\mathbf{x}}{\sqrt{k_2}} - \frac{\gamma \mathbf{y}}{2\sqrt{k_2}} \right) d\mathbf{y} \tag{4}$$

The choice of the scaling factors in the definition is crucial and characteristic of parabolic waves.

The Wigner distribution has the following properties:

$$\int W_z(\mathbf{x}, \mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{y}} d\mathbf{p} = \Psi_1 \left(z, \frac{\mathbf{x}}{\sqrt{k_1}} + \frac{\gamma \mathbf{y}}{2\sqrt{k_1}} \right) \Psi_2^* \left(z, \frac{\mathbf{x}}{\sqrt{k_2}} - \frac{\gamma \mathbf{y}}{2\sqrt{k_2}} \right) \tag{5}$$

$$\int W_z(\mathbf{x}, \mathbf{p}) e^{-i\mathbf{x}\cdot\mathbf{q}} d\mathbf{x} = \left(\frac{\sqrt{k_1 k_2} \pi^2}{\gamma} \right)^d \widehat{\Psi}_1 \left(z, \frac{\sqrt{k_1} \mathbf{p}}{4\gamma} + \frac{\sqrt{k_1} \mathbf{q}}{2} \right) \widehat{\Psi}_2^* \left(z, \frac{\sqrt{k_2} \mathbf{p}}{4\gamma} - \frac{\sqrt{k_2} \mathbf{q}}{2} \right) \tag{6}$$

and hence contains all the information in the two-point two-frequency function. Furthermore, the two-frequency Wigner distribution satisfies the two-frequency Wigner–Moyal equation exactly [6]

$$\frac{\partial W_z}{\partial z} + \mathbf{p} \cdot \nabla_{\mathbf{x}} W_z + \frac{1}{\sqrt{\alpha\delta}} \mathcal{V}_z W_z = 0 \tag{7}$$

with the initial condition W_0 where the operator \mathcal{V}_z is given as

$$\mathcal{V}_z W_z = i \int \theta^{-1} \left[e^{i\mathbf{q}\cdot\mathbf{x}/(\sqrt{k_1\varepsilon})} k_1 W_z \left(\mathbf{x}, \mathbf{p} + \frac{\theta \mathbf{q}}{2\sqrt{k_1}} \right) - e^{i\mathbf{q}\cdot\mathbf{x}/(\sqrt{k_2\varepsilon})} k_2 W_z \left(\mathbf{x}, \mathbf{p} - \frac{\theta \mathbf{q}}{2\sqrt{k_2}} \right) \right] \widehat{V} \left(\frac{\alpha z}{\varepsilon}, d\mathbf{q} \right)$$

2. Assumptions on the refractive index fluctuation

We assume that $V_z(\mathbf{x}) = V(z, \mathbf{x})$ is a centered, z -stationary, \mathbf{x} -homogeneous random field admitting the spectral representation

$$V_z(\mathbf{x}) = \int \exp(i\mathbf{p} \cdot \mathbf{x}) \widehat{V}_z(d\mathbf{p})$$

with the z -stationary spectral measure $\widehat{V}_z(\cdot)$ satisfying

$$\mathbb{E}[\widehat{V}_z(d\mathbf{p}) \widehat{V}_z(d\mathbf{q})] = \delta(\mathbf{p} + \mathbf{q}) \Phi_0(\mathbf{p}) d\mathbf{p} d\mathbf{q}$$

The transverse power spectrum density is related to the full power spectrum density $\Phi(w, \mathbf{p})$ as $\Phi_0(\mathbf{p}) = \int \Phi(w, \mathbf{p}) dw$. The power spectral density $\Phi(\mathbf{k})$ satisfies $\Phi(\mathbf{k}) = \Phi(-\mathbf{k})$, $\forall \mathbf{k} = (w, \mathbf{p}) \in \mathbb{R}^{d+1}$, because the electric susceptibility field is assumed to be real-valued. Hence $\Phi(w, \mathbf{p}) = \Phi(-w, \mathbf{p}) = \Phi(w, -\mathbf{p}) = \Phi(-w, -\mathbf{p})$ which is related to the detailed balance of the limiting scattering operators described below.

More specifically we make the following two assumptions:

Assumption 1. $V(z, \mathbf{x})$ is a Gaussian process with a full spectral density $\Phi(\mathbf{k})$, $\mathbf{k} \in \mathbb{R}^{d+1}$, which is uniformly bounded and decays at $|\mathbf{k}| = \infty$ with sufficiently high power of $|\mathbf{k}|^{-1}$.

We note that the assumption of Gaussianity is not essential and is made here to simplify the presentation.

Let \mathcal{F}_z and \mathcal{F}_z^+ be the sigma-algebras generated by $\{V_s: \forall s \leq z\}$ and $\{V_s: \forall s \geq z\}$, respectively and let $L^2(\mathcal{F}_z)$ and $L^2(\mathcal{F}_z^+)$ denote the square-integrable functions measurable w.r.t. to them respectively. The maximal correlation coefficient $r(t)$ is given by

$$\rho(t) = \sup_{\substack{h \in L^2(\mathcal{F}_z) \\ \mathbb{E}[h]=0, \mathbb{E}[h^2]=1}} \sup_{\substack{g \in L^2(\mathcal{F}_{z+t}^+) \\ \mathbb{E}[g]=0, \mathbb{E}[g^2]=1}} \mathbb{E}[hg] \tag{8}$$

Assumption 2. The maximal correlation coefficient $\rho(t)$ is integrable: $\int_0^\infty \rho(s) ds < \infty$.

3. Main theorems

Theorem 1. [7] *Let $\theta > 0$ be fixed. Then under the two-frequency radiative transfer scaling (2)–(3) the weak solutions of the Wigner–Moyal equation (7) converge in law in the space $C([0, \infty), L_w^2(\mathbb{R}^d))$ to that of the following deterministic equation*

$$\frac{\partial}{\partial z} W + \mathbf{p} \cdot \nabla W = \frac{2\pi k^2}{\theta^2} \int K(\mathbf{p}, \mathbf{q}) \left[e^{-i\beta\theta\mathbf{q}\cdot\mathbf{x}/(2k^{1/2})} W\left(\mathbf{x}, \mathbf{p} + \frac{\theta\mathbf{q}}{\sqrt{k}}\right) - W(\mathbf{x}, \mathbf{p}) \right] d\mathbf{q} \tag{9}$$

with the same initial condition W_0 where the kernel K is given by

$$K(\mathbf{p}, \mathbf{q}) = \frac{1}{\alpha} \Phi\left(\alpha^{-1}\left(\mathbf{p} + \frac{\theta\mathbf{q}}{2\sqrt{k}}\right) \cdot \mathbf{q}, \mathbf{q}\right) \tag{10}$$

Here and below $L_w^2(\mathbb{R}^{2d})$ is the space of complex-valued square integrable functions on the phase space \mathbb{R}^{2d} endowed with the weak topology.

Remark 1. If we now let $\alpha \rightarrow 0$, then the kernel becomes

$$K(\mathbf{p}, \mathbf{q}) = \delta\left(\left(\mathbf{p} + \frac{\theta\mathbf{q}}{2\sqrt{k}}\right) \cdot \mathbf{q}\right) \int \Phi(w, \mathbf{q}) dw$$

We refer to this as the transverse case because the transverse correlation length ε is much shorter than the longitudinal correlation length δ .

On the other hand, in the longitudinal case $\alpha \rightarrow \infty$ the limiting kernel would vanish. In order to maintain an interesting limit, we increase μ by a factor of $\sqrt{\alpha}$. Then the kernel for the longitudinal case becomes

$$K(\mathbf{p}, \mathbf{q}) = \Phi(0, \mathbf{q}) \quad (11)$$

In both the longitudinal and transverse cases the fluctuations in the refractive index are extremely anisotropic.

Theorem 2. [7] Assume $\lim_{\varepsilon \rightarrow 0} \theta = 0$. Then under the two-frequency radiative transfer scaling (2)–(3) the weak solutions of the Wigner–Moyal equation (7) converge in law in the space $C([0, \infty), L_w^2(\mathbb{R}^d))$ to that of the following deterministic equation

$$\frac{\partial}{\partial z} W + \mathbf{p} \cdot \nabla W = k \left(\nabla_{\mathbf{p}} + \frac{i\beta}{2} \mathbf{x} \right) \cdot \mathbf{D} \cdot \left(\nabla_{\mathbf{p}} + \frac{i\beta}{2} \mathbf{x} \right) W(\mathbf{x}, \mathbf{p}) \quad (12)$$

with the same initial condition W_0 where the (momentum) diffusion coefficient \mathbf{D} is given by

$$\mathbf{D}(\mathbf{p}) = \frac{\pi}{\alpha} \int \Phi \left(\frac{\mathbf{p} \cdot \mathbf{q}}{\alpha}, \mathbf{q} \right) \mathbf{q} \otimes \mathbf{q} \, d\mathbf{q} \quad (13)$$

Remark 2. In the transverse case $\alpha \rightarrow 0$, the limiting coefficient is

$$\mathbf{D}(\mathbf{p}) = \pi |\mathbf{p}|^{-1} \int_{\mathbf{p} \cdot \mathbf{p}_{\perp} = 0} \int \Phi(w, \mathbf{p}_{\perp}) \, dw \, \mathbf{p}_{\perp} \otimes \mathbf{p}_{\perp} \, d\mathbf{p}_{\perp} \quad (14)$$

For the longitudinal case $\alpha \rightarrow \infty$, we increase μ by a factor of $\sqrt{\alpha}$ as before such that the limiting coefficient is nontrivial

$$\mathbf{D} = \pi \int \Phi(0, \mathbf{q}) \mathbf{q} \otimes \mathbf{q} \, d\mathbf{q} \quad (15)$$

When $k_1 = k_2$ or $\beta = 0$, Eqs. (9) and (12) reduce to the standard radiative transfer equations derived in [2,3]. The proof of these results follows exactly the strategy developed in [2] and outlined in [3], originally developed for the standard one-frequency Wigner distribution (see [6] for the same strategy applied to the two-frequency Wigner distribution for a different scaling limit).

Another notable fact is that Eq. (9) with (11) and Eq. (12) with (15) are similar to the governing equations for the ensemble-averaged two-frequency Wigner distribution for the z -white-noise potential analyzed in [6]. This can be understood by the similar behaviors of the potential more rapidly fluctuating in z to the z -white-noise potential and the (less) rapid fluctuation in \mathbf{x} gives rise to self-averaging which is lacking in the z -white-noise potential.

4. The longitudinal and transverse case

To illustrate the utility of these equations, we proceed to discuss the two special cases for the transverse dimension $d = 2$. For simplicity, we will assume the isotropy of the medium in the transverse coordinates such that $\Phi(w, \mathbf{p}) = \Phi(w, |\mathbf{p}|)$. As a consequence the momentum diffusion coefficient is a scalar. In the longitudinal case $\mathbf{D} = D\mathbf{I}$ with a constant scalar D whereas in the transverse case $\mathbf{D}(\mathbf{p}) = C|\mathbf{p}|^{-1} \hat{\mathbf{p}}_{\perp} \otimes \hat{\mathbf{p}}_{\perp}$ with the constant C given by

$$C = \frac{\pi}{2} \iint \Phi(w, \mathbf{p}_{\perp}) \, dw |\mathbf{p}_{\perp}|^2 \, d\mathbf{p}_{\perp}$$

Here $\hat{\mathbf{p}}_{\perp} \in \mathbb{R}^2$ is a unit vector normal to $\mathbf{p} \in \mathbb{R}^2$.

First of all, Eq. (12) by itself gives qualitative information about three important parameters of the stochastic channel: the spatial spread σ_* , the coherence length ℓ_c and the coherence bandwidth β_c , through the following scaling argument. One seeks the change of variables

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{\sigma_* \sqrt{k}}, \quad \tilde{\mathbf{p}} = \mathbf{p} \ell_c \sqrt{k}, \quad \tilde{z} = \frac{z}{L}, \quad \tilde{\beta} = \frac{\beta}{\beta_c} \quad (16)$$

where L is the propagation distance to remove all the physical parameters from (12) and to aim for the form

$$\frac{\partial}{\partial \tilde{z}} W + \tilde{\mathbf{p}} \cdot \nabla_{\tilde{\mathbf{x}}} W = \left(\nabla_{\tilde{\mathbf{p}}} + \frac{i\tilde{\beta}}{2} \tilde{\mathbf{x}} \right) \cdot \left(\nabla_{\tilde{\mathbf{p}}} + \frac{i\tilde{\beta}}{2} \tilde{\mathbf{x}} \right) W \quad (17)$$

in the longitudinal case and the form

$$\frac{\partial}{\partial \tilde{z}} W + \tilde{\mathbf{p}} \cdot \nabla_{\tilde{\mathbf{x}}} W = \left(\nabla_{\tilde{\mathbf{p}}} + \frac{i\tilde{\beta}}{2} \tilde{\mathbf{x}} \right) \cdot \frac{\hat{\mathbf{p}}_{\perp} \otimes \hat{\mathbf{p}}_{\perp}}{|\hat{\mathbf{p}}|} \cdot \left(\nabla_{\tilde{\mathbf{p}}} + \frac{i\tilde{\beta}}{2} \tilde{\mathbf{x}} \right) W \quad (18)$$

in the transverse case. From the left side of (12) it immediately follows the first duality relation $\ell_c \sigma_* \sim L/k$. The balance of terms inside each pair of parentheses leads to the second duality relation $\beta_c \sim \ell_c / \sigma_*$. Finally the removal of D or C determines the spatial spread σ_* which has a different expression in the longitudinal and transverse case. In the longitudinal case,

$$\sigma_* \sim D^{1/2} L^{3/2}, \quad \ell_c \sim k^{-1} D^{-1/2} L^{-1/2}, \quad \beta_c \sim D^{-1} L^{-2}$$

whereas in the transverse case

$$\sigma_* \sim k^{-1/6} C^{1/3} L^{4/3}, \quad \ell_c \sim k^{-5/6} C^{-1} L^{-1}, \quad \beta_c \sim k^{-2/3} C^{-2/3} L^{-5/3}$$

In the longitudinal case, the inverse Fourier transform in $\tilde{\mathbf{p}}$ renders Eq. (17) to the form

$$\frac{\partial \tilde{W}}{\partial \tilde{z}} - i \nabla_{\tilde{\mathbf{y}}} \cdot \nabla_{\tilde{\mathbf{x}}} \tilde{W} = - \left| \tilde{\mathbf{y}} - \frac{\tilde{\beta}}{2} \tilde{\mathbf{x}} \right|^2 \tilde{W} \quad (19)$$

which can be solved exactly and whose Green function at $\tilde{z} = 1$ is [8]

$$\begin{aligned} & \frac{(1+i)^{d/2} \tilde{\beta}^{d/4}}{(2\pi)^d \sin^{d/2} [\tilde{\beta}^{1/2}(1+i)]} \exp \left[i \frac{|\tilde{\mathbf{y}} - \mathbf{y}'|^2}{2\tilde{\beta}} \right] \exp \left[i \frac{(\tilde{\mathbf{y}} - \mathbf{y}') \cdot (\tilde{\mathbf{x}} - \mathbf{x}')}{2} \right] \exp \left[i \frac{\tilde{\beta} |\tilde{\mathbf{x}} - \mathbf{x}'|^2}{8} \right] \\ & \times \exp \left[\frac{1-i}{2\tilde{\beta}^{1/2}} \cot(\tilde{\beta}^{1/2}(1+i)) \left| \tilde{\mathbf{y}} - \tilde{\beta} \tilde{\mathbf{x}}/2 - \frac{\mathbf{y}' - \tilde{\beta} \mathbf{x}'/2}{\cos(\tilde{\beta}^{1/2}(1+i))} \right|^2 \right] \\ & \times \exp \left[-\frac{1-i}{2\tilde{\beta}^{1/2}} |\mathbf{y}' - \tilde{\beta} \mathbf{x}'/2|^2 \tan(\tilde{\beta}^{1/2}(1+i)) \right] \end{aligned}$$

This solution gives asymptotically precise information about the cross-frequency correlation, important for analyzing the information transfer and time reversal with broadband signals in the channel described by the random Schrödinger equation [8] (see also [9–11]). It is unclear if the transverse case is exactly solvable or not.

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