Compressed Sensing Opportunities in SAR?

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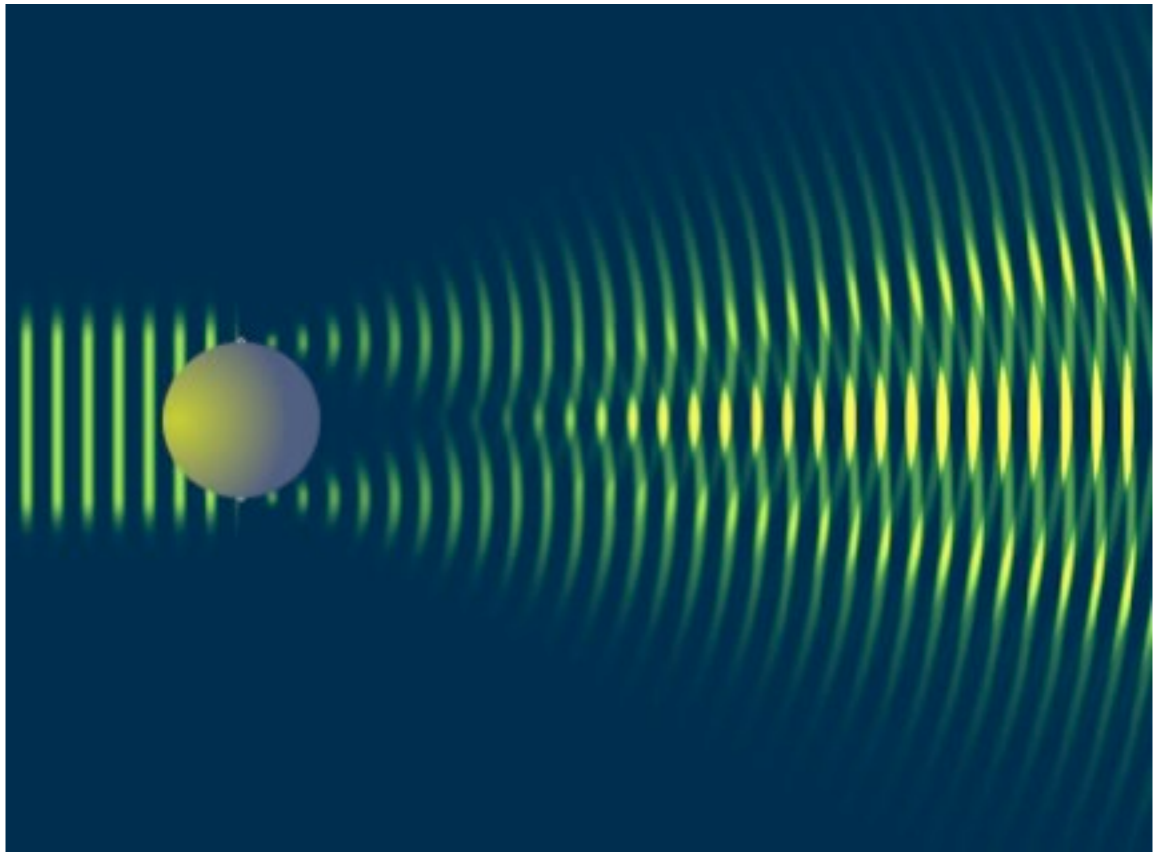
Monochromatic Scattering

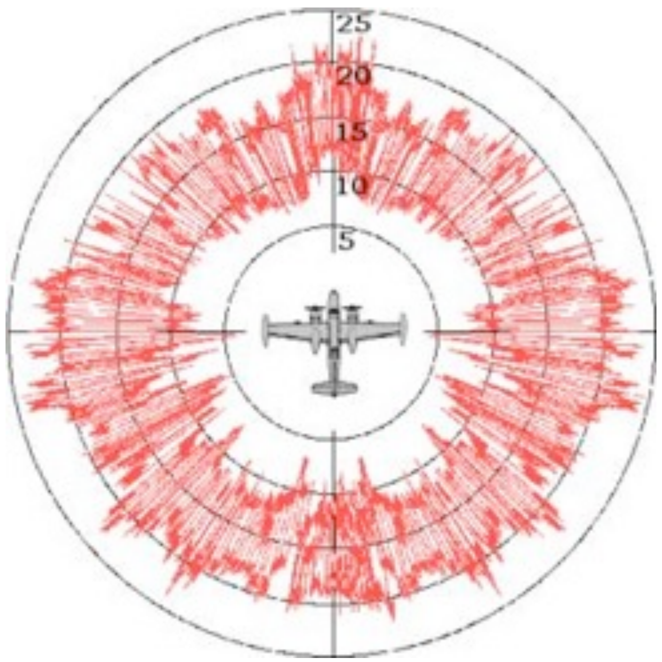


G : Green function for $-\Delta - \omega^2$.

Lippmann-Schwinger equation:

$$u^{\mathrm{s}}(\mathbf{r}) = \omega^2 \int_{\mathbb{R}^d} V(\mathbf{r}') \left(u^{\mathrm{i}}(\mathbf{r}') + u^{\mathrm{s}}(\mathbf{r}') \right) G(\mathbf{r}, \mathbf{r}') d\mathbf{r}'$$

Far-Field Back-Scattering Measurement



$\hat{d}$ Incident direction

$\hat{r}$ Receptive direction

$$u^{\mathrm{s}}(\mathbf{r}) = \frac{e^{i\omega|\mathbf{r}|}}{|\mathbf{r}|^{(d-1)/2}}\left(A(\hat{\mathbf{r}},\hat{\mathbf{d}}) + \mathcal{O}\left(\frac{1}{|\mathbf{r}|}\right)\right), \quad \hat{\mathbf{r}}$$

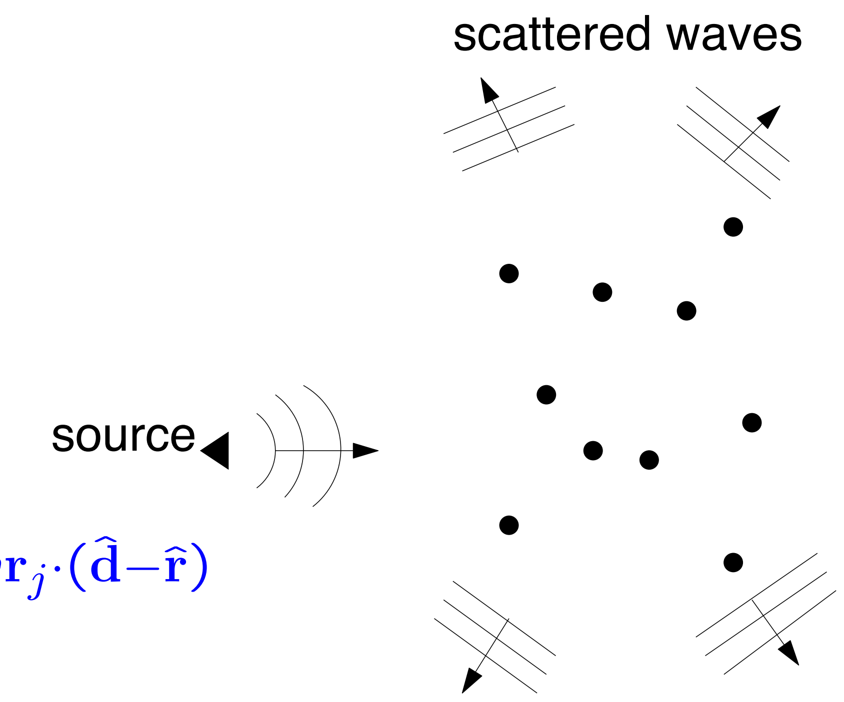
$$A(\hat{\mathbf{r}},\hat{\mathbf{d}}) = \frac{\omega^{\frac{(d+1)}{2}}}{4\pi}\int_{\mathbb{R}^d} V(\mathbf{r}')u(\mathbf{r}')e^{-i\omega\mathbf{r}'\cdot\hat{\mathbf{r}}}d\mathbf{r}'$$

RCS $\sigma = |A(-\hat{\mathbf{d}},\hat{\mathbf{d}})|^2$ Far-field SAR

Point targets

- **Scattering amplitude is a finite sum**

$$
\begin{aligned}
A(\hat{\mathbf{r}}, \hat{\mathbf{d}}) &= \frac{\omega^2}{4\pi} \sum_{j=1}^{m} V_j u(\mathbf{r}_j) e^{-\mathrm{i}\omega \mathbf{r}_j \cdot \hat{\mathbf{r}}} \\
u(\mathbf{r}_i) &= u^{\mathrm{i}}(\mathbf{r}_i) + \omega^2 \sum_{i \neq j} G(\mathbf{r}_i, \mathbf{r}_j) V_j u(\mathbf{r}_j)
\end{aligned}
$$



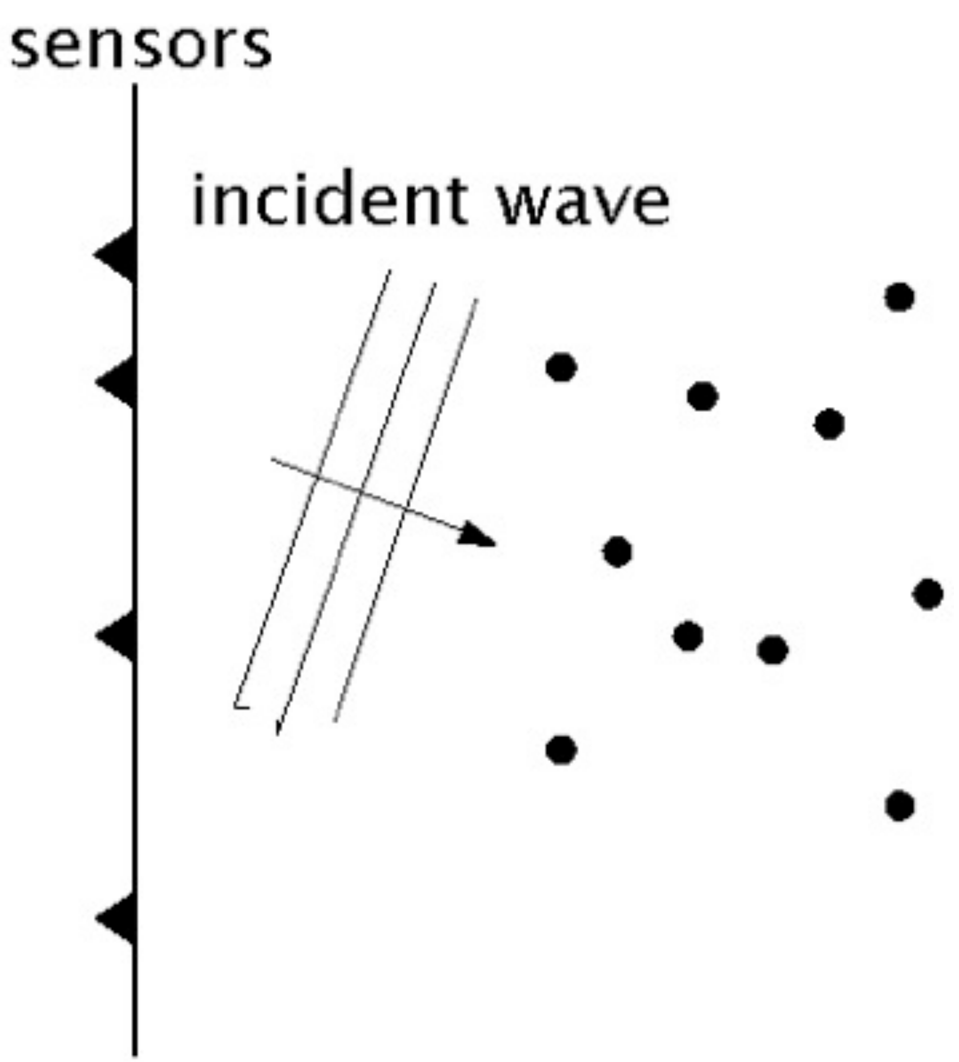
- **Born approximation,**

$$
A(\hat{\mathbf{r}}, \hat{\mathbf{d}}) = \frac{\omega^2}{4\pi} \sum_{j=1}^{m} V_j \mathrm{e}^{\mathrm{i}\omega \mathbf{r}_j \cdot \hat{\mathbf{d}}} e^{-i\omega \mathbf{r}_j \cdot \hat{\mathbf{r}}} = \frac{\omega^2}{4\pi} \sum_{j=1}^{m} V_j e^{\mathrm{i}\omega \mathbf{r}_j \cdot (\hat{\mathbf{d}} - \hat{\mathbf{r}})}
$$

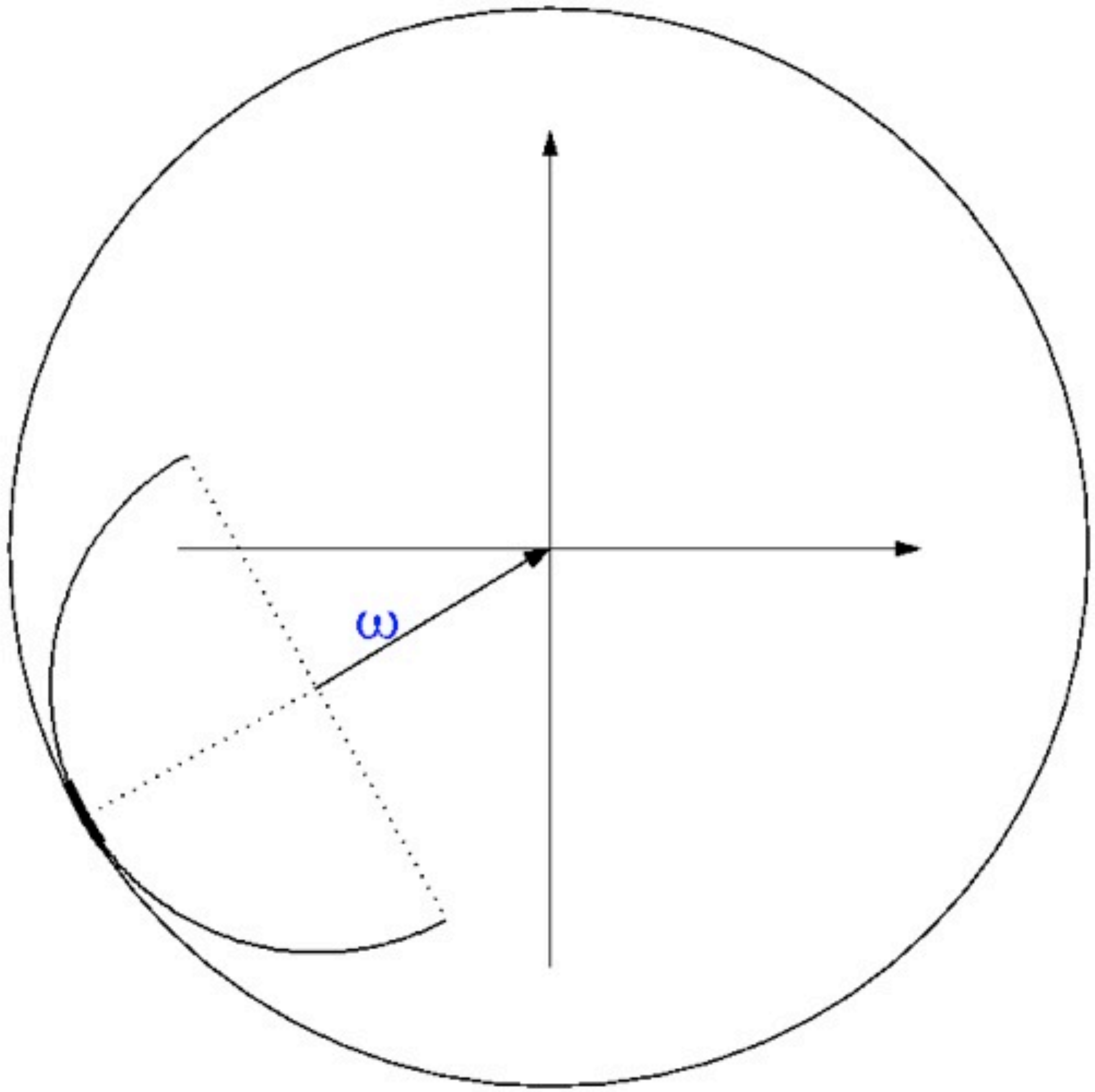
- **Back-scattering sampling:** $\hat{\mathbf{r}} = -\hat{\mathbf{d}}$

$$
A(\hat{\mathbf{r}}, \hat{\mathbf{d}}) = \frac{\omega^2}{4\pi} \sum_{j=1}^{m} V_j e^{2\mathrm{i}\omega \mathbf{r}_j \cdot \hat{\mathbf{d}}}
$$

Diffraction Tomography



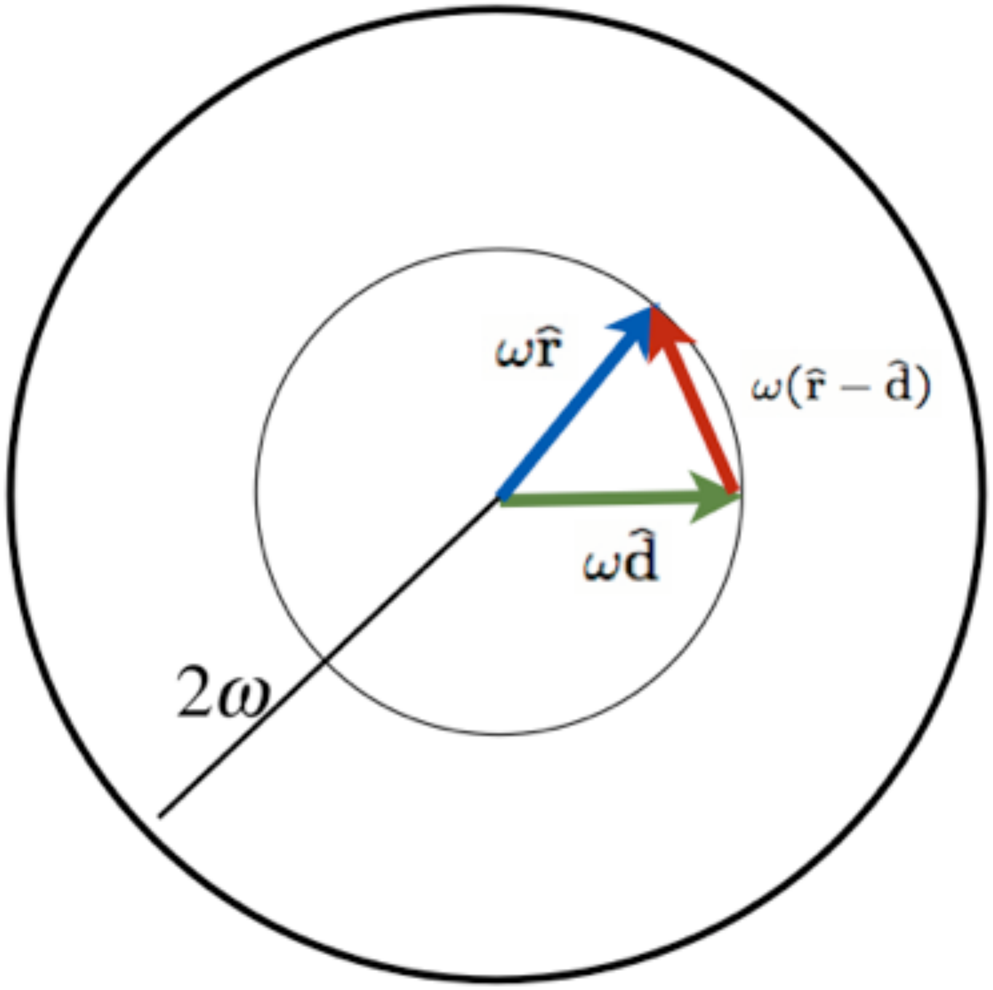
Near field



Fourier sampling

Fourier sampling

- Fourier sampling by moving sensor around



Compressed Sensing

$$Y = \Phi X + E, \quad Y \in \mathbb{C}^M, \; X \in \mathbb{C}^N$$

$$E = \text{model error} + \text{external error}$$

$$M \ll N$$

Candes ... Donoho ... Romberg ... Tao ... Tropp ....

- Sparsity: bases/dictionaries
- Random measurements: incoherence, RIP
- Algorithms: L1-min, greedy (OMP)

$$\min \|Z\|_1, \quad s.t. \; \|Y - \Phi Z\|_2 \leq \|E\|_2$$

- Discrete: grid of size $\ell \longrightarrow$ gridding error
- Linear: Born approximation

Agenda

- How do random schemes help ?
- How to discretize & sparsify?
- How to handle coherent sensing matrix?
- How to handle multiple scattering?

Performance Predictors

Mutual coherence: peak sidelobe

$$\mu(\Phi) = \max_{i \neq j} \frac{\left|\sum_k \Phi_{ik}^* \Phi_{kj}\right|}{\sqrt{\sum_k |\Phi_{ki}|^2}\sqrt{\sum_k |\Phi_{kj}|^2}}$$

CS can recover $\sim 1/\mu$ objects

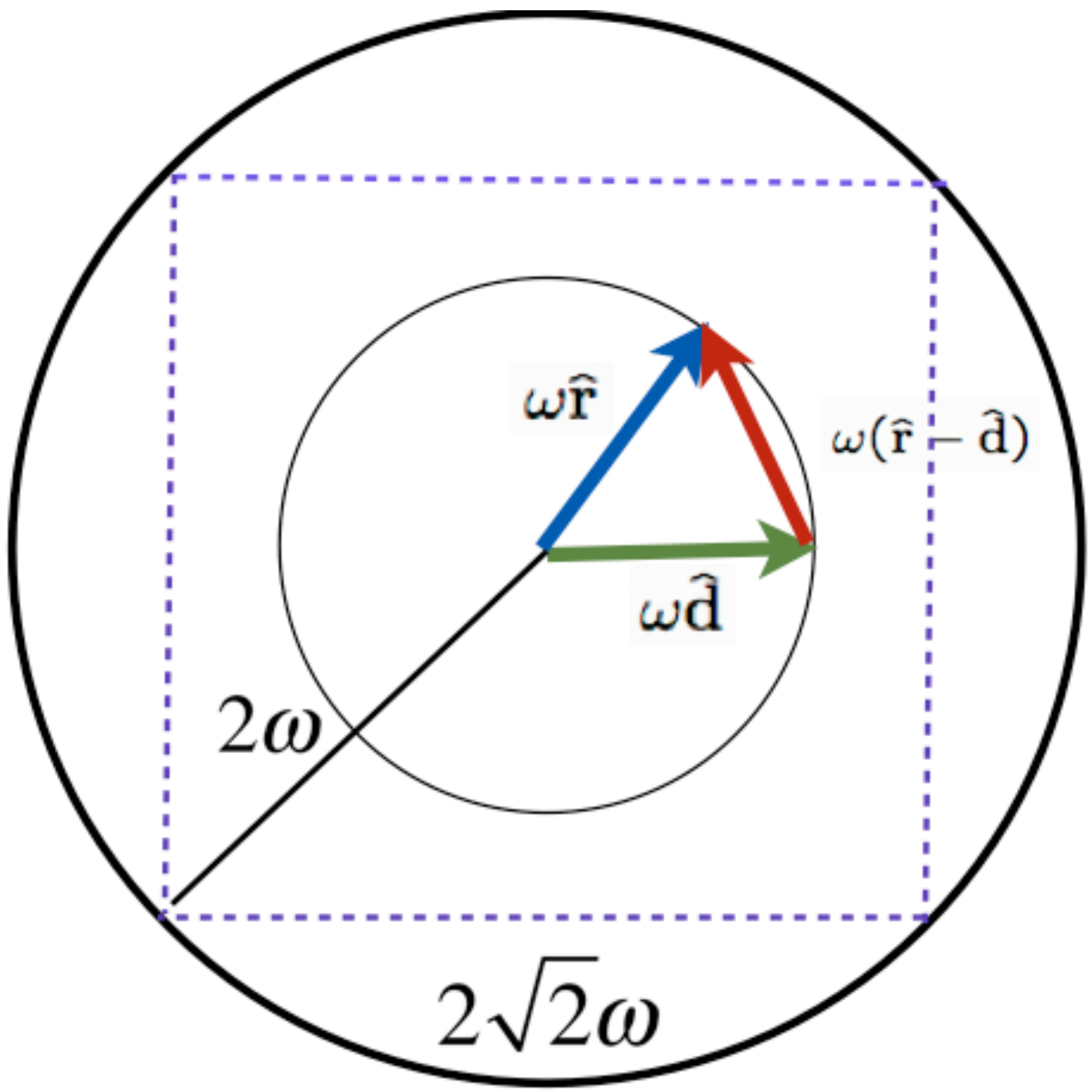
Restricted isometry property (RIP)

$s \log N \times N$ random partial Fourier matrix

CS can recover s targets

Fourier Cell

Resolution: $\dfrac{\lambda}{2\sqrt{2}}$



CS-SAR

- $(\rho_l, \phi_l), l = 1, .., M$ **polar** **coordinates of** **i.i.d.** **uniform** **r.v.s** $(\xi_l, \eta_l) \in [0, 2\pi]^2$.

- $\Omega-$**band limited probes, i.e.** $\omega_l \in [0, \Omega]$. **Set**

$$\begin{aligned} \tilde{\theta}_l &= \theta_l + \pi = \phi_l \quad \text{(backscattering sampling)} \\ \omega_l &= \frac{\Omega \rho_l}{\sqrt{2}} \end{aligned}$$

$l = 1, ..., M$.

THEOREM

Suppose

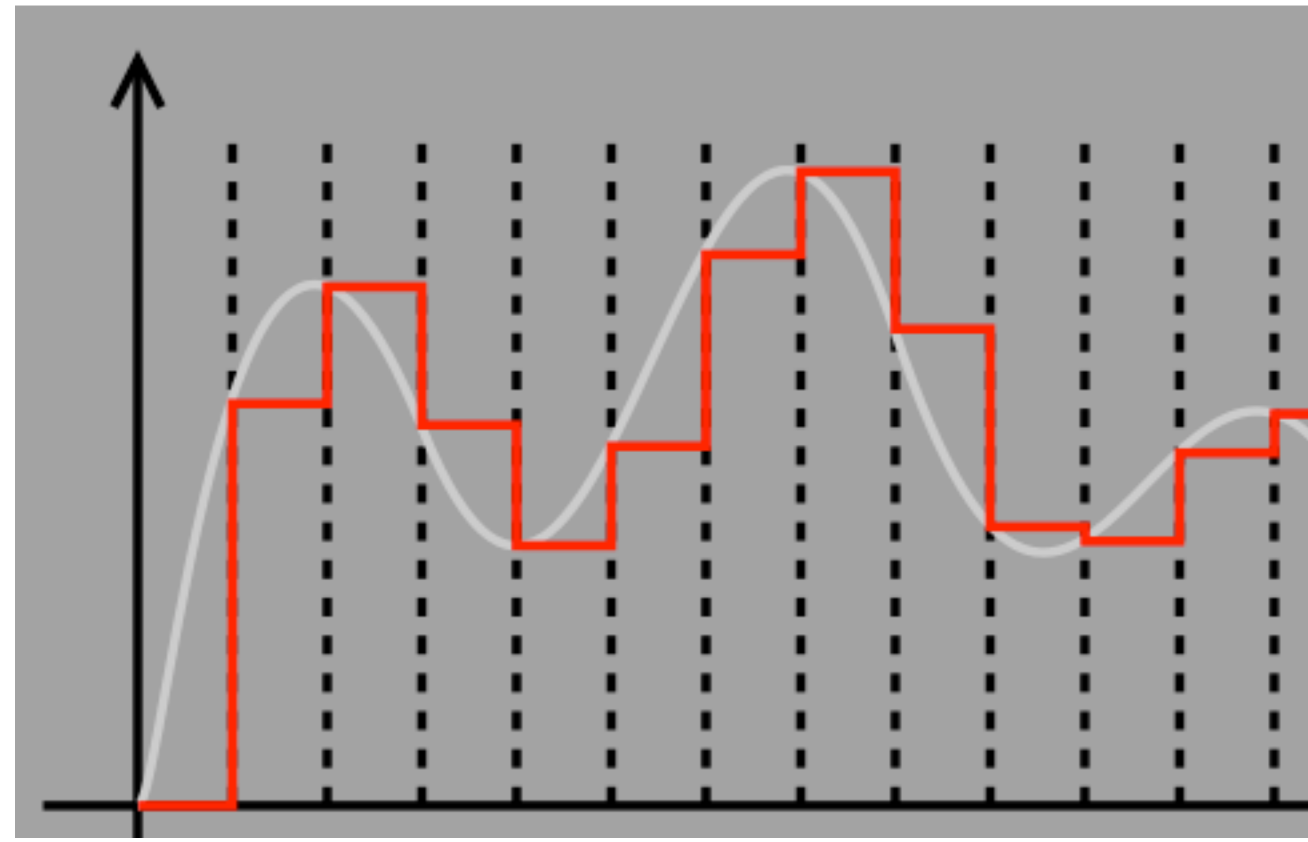
$$\Omega \ell = \pi / \sqrt{2}.$$

Then with high probability the L^1**-minimizer satisfies the error bound**

$$\|\hat{X} - X\|_2 \leq C_1 s^{-1/2} \|X - X^{(s)}\|_1 + C_2 \varepsilon.$$

Linear (LFM) or quadratic chirp: random time samples

Extended Object



piecewise constant approximation

$$V \rightarrow V_\ell$$

$$E = E_{disc} + E_{ext}$$

Discretization Error

point object does not interfere with itself,
but a pixel/voxel does.

$$\hat{g}(k_1,k_2) = \frac{2}{\pi}\cdot\frac{\sin\frac{k_1}{2}}{k_1}\cdot\frac{\sin\frac{k_2}{2}}{k_2}$$

$$\|E\|_\infty \le \frac{\|V - V_\ell\|_{L^1}}{2\pi \min_l |\hat{g}(\ell\omega_l(\hat{\mathbf{d}}_l - \hat{\mathbf{r}}_l))|}$$

To avoid small divisor, need $\ell < \lambda/2$

multi-shot SISO $\ell = \dfrac{\lambda}{2\sqrt{2}}$

$$\|V - V_\ell\|_{L^1} \le \frac{4}{\pi^2}\varepsilon \Longrightarrow \|E\|_2 \le \varepsilon$$

Sparse Extended Object

THEOREM (F 09) For CS-SAR sampling scheme with $M = \mathcal{O}(s \log N)$, if

$$\|V - V_\ell\|_1 \leq \frac{4\varepsilon}{\pi^2}$$

then with high probability

$$\|\hat{X} - X\|_2 \leq C_1 s^{-1/2} \|X - X^{(s)}\|_1 + C_2 \varepsilon$$

40 X 80 original, sparsity= 161, #data= 500

Distributed extended targets

- **Wavelet expansion**

$$V(x,z) = \sum_{\mathbf{p},\mathbf{q}\in\mathbb{Z}^2} V_{\mathbf{p},\mathbf{q}}\psi_{\mathbf{p},\mathbf{q}}(x,z)$$

where

$$\psi_{\mathbf{p},\mathbf{q}}(\mathbf{r}) = 2^{-(p_1+p_2)/2}\psi(2^{-\mathbf{p}}\mathbf{r}-\mathbf{q}), \qquad \mathbf{p},\mathbf{q}\in\mathbb{Z}^2$$

with

$$2^{-\mathbf{p}}\mathbf{r} = (2^{-p_1}x, 2^{-p_2}z)$$

form an ONB in $L^2(\mathbb{R}^2)$.

- **Littlewood-Paley basis**

$$\psi(\mathbf{r}) = (\pi^2 xz)^{-1}(\sin(2\pi x)-\sin(\pi x))\cdot(\sin(2\pi z)-\sin(\pi z))$$

which is band-limited

$$\widehat{\psi}(\xi,\zeta) = \begin{cases} (2\pi)^{-1}. & \pi\le|\xi|,|\zeta|\le 2\pi \\ 0, & \textbf{otherwise}. \end{cases}$$

Sensing matrix

- Incident field

$$u^{\mathrm{i}}_k(\mathbf{r}) = e^{i\omega_k \mathbf{r}\cdot\hat{\mathbf{d}}_k}, \quad k = 1, ..., M$$

we have

$$Y_k = 2\pi \sum_{\mathbf{p},\mathbf{q}\in\mathbb{Z}^2} 2^{(p_1+p_2)/2} V_{\mathbf{p},\mathbf{q}} e^{i\omega_k 2^{\mathbf{p}}(\hat{\mathbf{d}}_k - \hat{\mathbf{r}}_k)\cdot\mathbf{q}} \hat{\psi}(\omega_k 2^{\mathbf{p}}(\hat{\mathbf{r}}_k - \hat{\mathbf{d}}_k))$$

with cutoffs

$$|\mathbf{q}|_\infty \le m_{\mathbf{p}}, \quad |\mathbf{p}|_\infty \le p_*, \quad |\mathbf{q}'|_\infty \le n_{\mathbf{p}'}, \quad |\mathbf{p}'|_\infty \le p_*.$$

- Sensing matrix

$$\Phi_{k,l} = \frac{1}{2n_{\mathbf{p}}+1} \hat{\psi}(\omega_k 2^{\mathbf{p}}(\hat{\mathbf{r}}_k - \hat{\mathbf{d}}_k)) e^{i\omega_k 2^{\mathbf{p}}(\hat{\mathbf{d}}_k - \hat{\mathbf{r}}_k)\cdot\mathbf{q}}$$

and let $\Phi = [\Phi_{k,l}]$, where $\hat{\mathbf{d}}_k, \hat{\mathbf{r}}_k, \omega_k$ are given below.

- Target vector

$$X_l = 2\pi(2n_{\mathbf{p}}+1)2^{(p_1+p_2)/2} V_{\mathbf{p},\mathbf{q}}$$

Sampling scheme

- **Sampling scheme:**

  Let ξ_k, ζ_k be independent, uniform random variables on $[-1, 1]$ and define

$$\alpha_k = \frac{\pi}{\omega_k 2^{p'_1}} \cdot \begin{cases} 1+\xi_k, & \xi_k \in [0,1] \\ -1+\xi_k, & \xi_k \in [-1,0] \end{cases}$$

$$\beta_k = \frac{\pi}{\omega_k 2^{p'_2}} \cdot \begin{cases} 1+\zeta_k, & \zeta_k \in [0,1] \\ -1+\zeta_k, & \zeta_k \in [-1,0] \end{cases}.$$

  Let (ρ_k, ϕ_k) be the polar coordinates of (α_k, β_k) for CS-SAR.

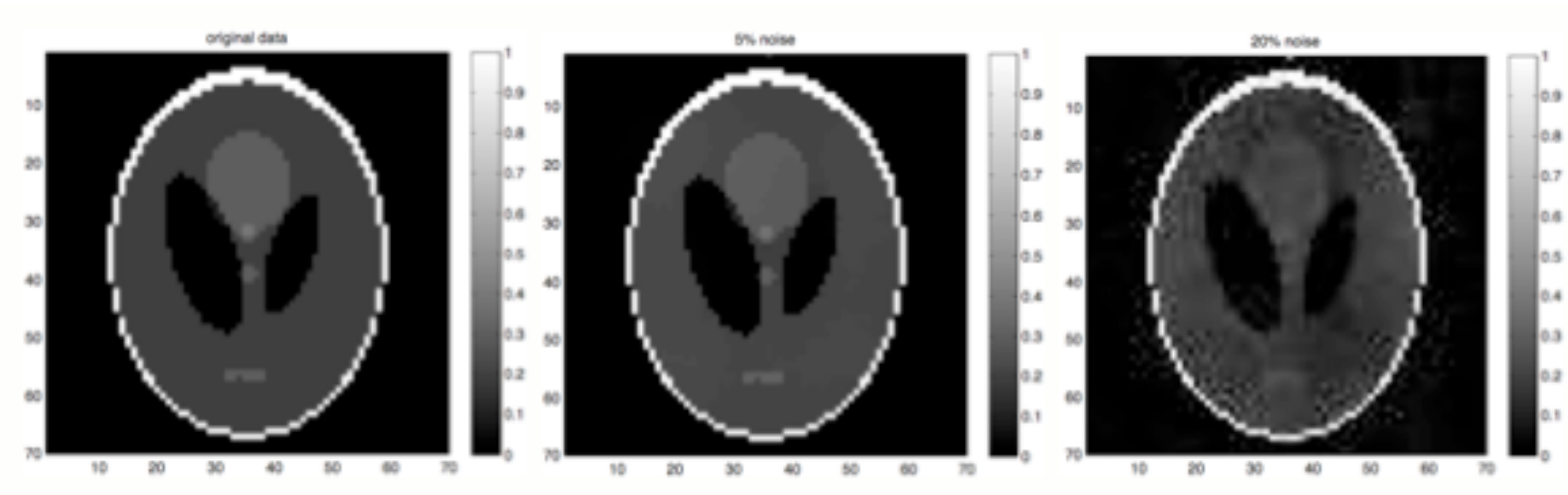
- $\Phi_{k,l}$ are zero if $\mathrm{p} \neq \mathrm{p}'$. Consequently the sensing matrix is the block-diagonal matrix with each block (indexed by $\mathrm{p} = \mathrm{p}'$) in the form of random Fourier matrix

$$\Phi_{k,l} = \frac{1}{2n_{\mathrm{p}}+1} e^{i\pi(q_1\xi_k + q_2\zeta_k)}.$$

  The above observation means that the target structures of different dyadic scales are decoupled and can be determined separately by our approach using compressed sensing techniques.

Piecewise Smooth Objects

TV min $$\min \|f\|_{TV}, \quad \text{s.t. } \|\Psi \nabla_d f - Z\|_2 \le \varepsilon.$$



70 X 70 original, sparsity of gradient= 836, #data=1000

Multiple-scattering

$$A(\hat{\mathbf{r}}_l, \hat{\mathbf{d}}) = \frac{\omega^2}{4\pi} \sum_{j=1}^{m} V_j u(\mathbf{r}_j) e^{-i\omega \mathbf{r}_j \cdot \hat{\mathbf{r}}_l}.$$

$$X = (V_j u(\mathbf{r}_j)) \in \mathbb{C}^m$$

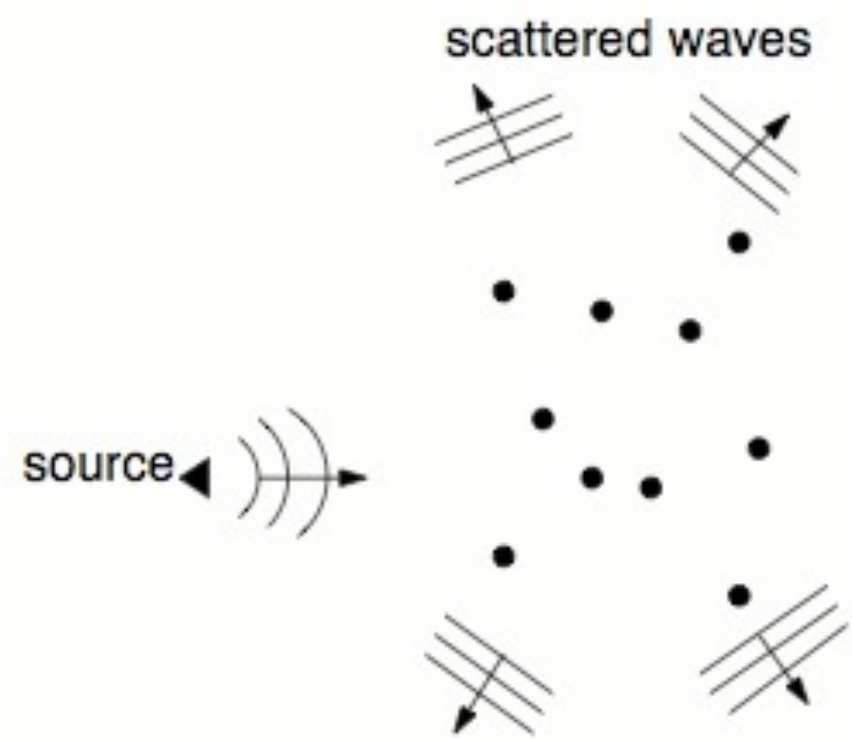
$$Y = 4\pi\omega^{-2}(A(\hat{\mathbf{r}}_l, \hat{\mathbf{d}}))$$

$$\phi_{lj} = e^{-i\omega(z_j \sin\tilde{\theta}_l + x_j \cos\tilde{\theta}_l)}, \qquad \mathbf{r}_j = (x_j, z_j)$$

$$Y = \Phi X + E$$
$$E = E_{grid} + E_{ext}$$

Single-Input-Multiple-Output (SIMO)

$$A(\hat{\mathbf{r}}_l, \hat{\mathbf{d}}) = \frac{\omega^2}{4\pi} \sum_{j=1}^{m} V_j u(\mathbf{r}_j) e^{-i\omega \mathbf{r}_j \cdot \hat{\mathbf{r}}_l}.$$



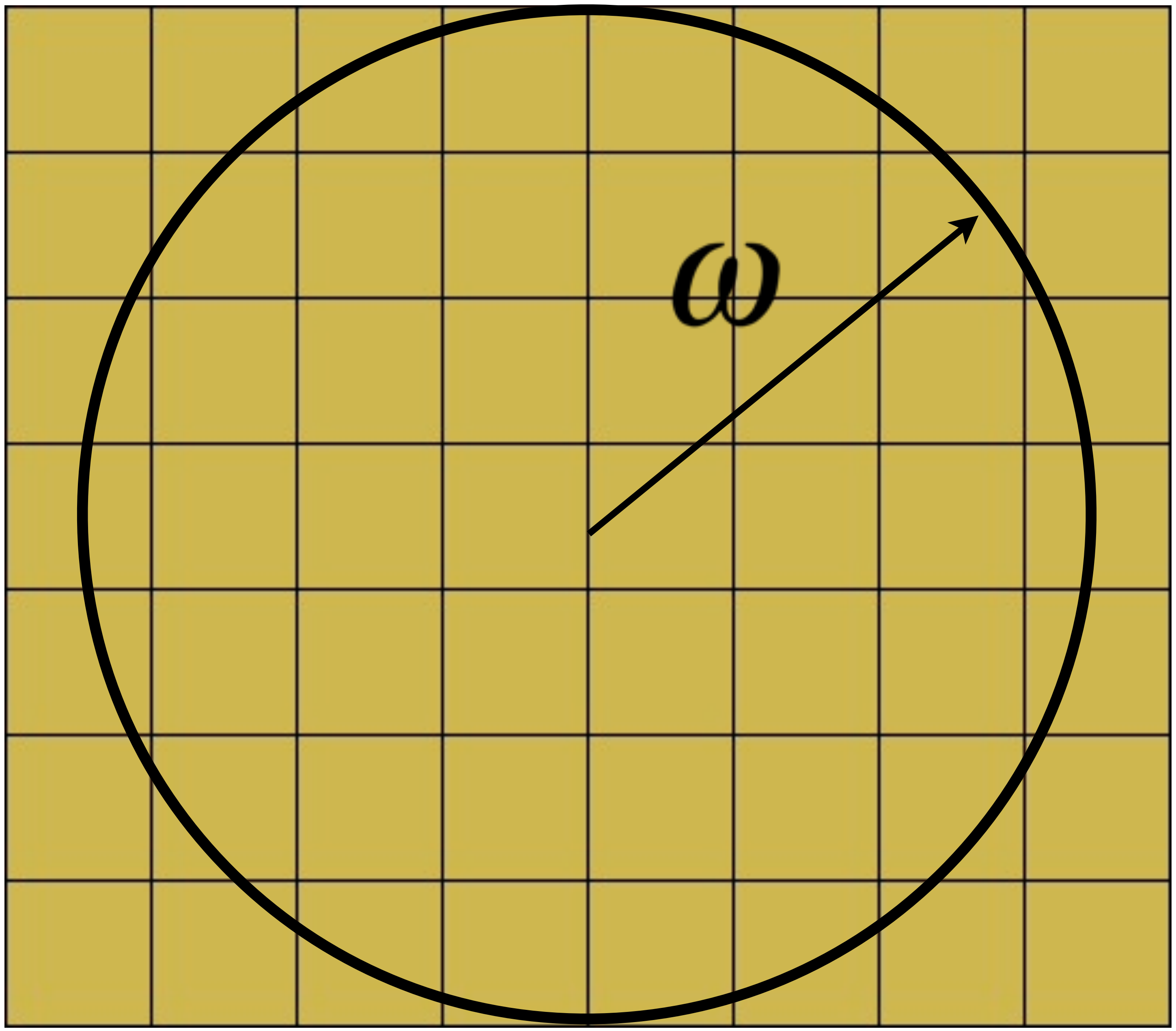
Foldy-Lax eq

$$u(\mathbf{r}_i) = u^{\mathrm{i}}(\mathbf{r}_i) + \omega^2 \sum_{i \neq j} G(\mathbf{r}_i, \mathbf{r}_j) V_j u(\mathbf{r}_j)$$

Assumption: point objects located on a finite regular grid of spacing ℓ.

Random sampling: the scattering directions $\hat{\mathbf{r}}_l, l = 1, ..., n$ are **i.i.d.**
$d = 2$: $\tilde{\theta}_l$ sampling angles

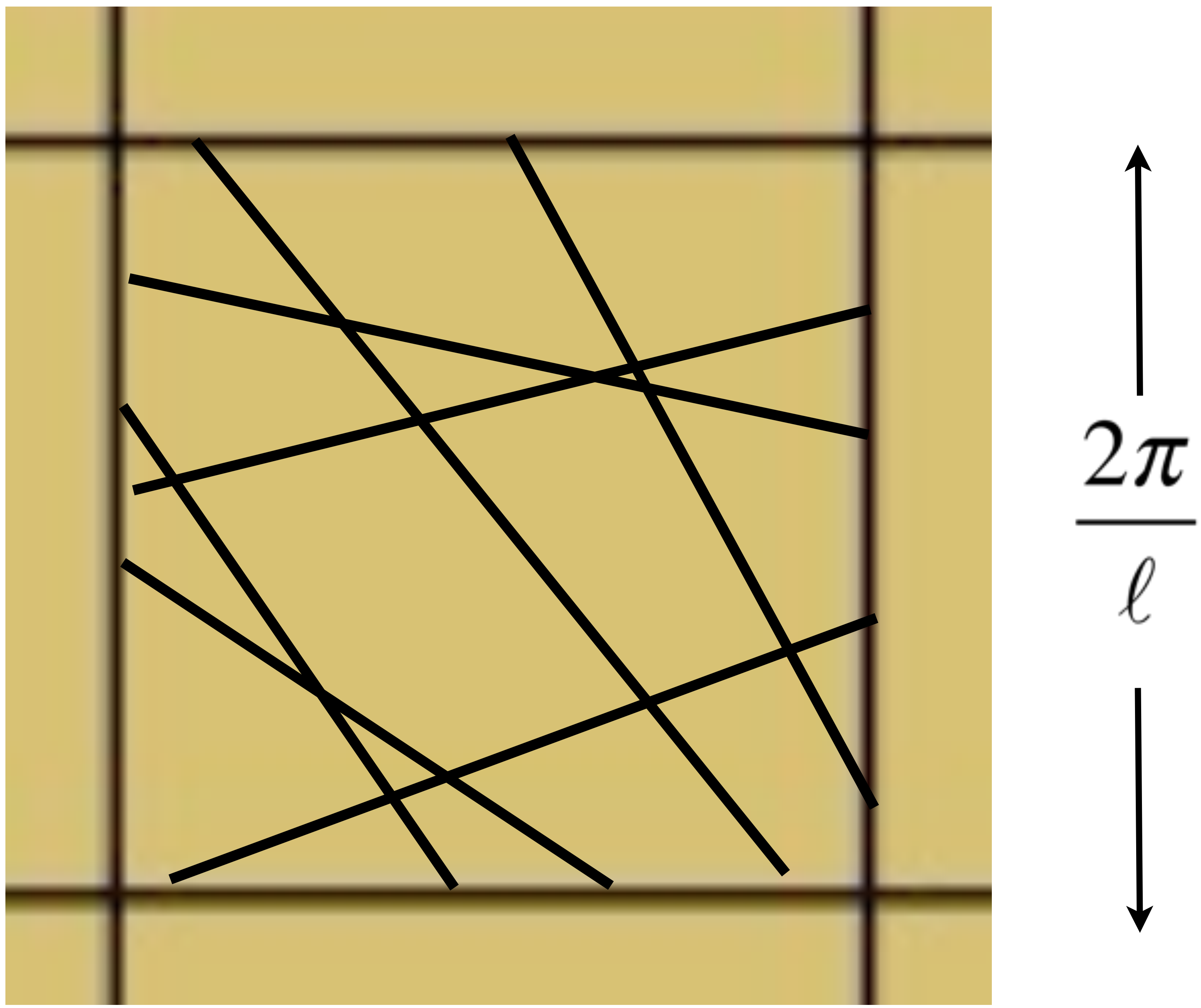
Backscattering Sampling



Fourier tiling of size $2\pi / \ell$

ℓ spatial resolution

High-frequency limit



$$\omega\ell \gg 1 \longrightarrow \ell \gg \lambda, \lambda = 2\pi / \omega$$

Coherence Bound

THEOREM (F 09) **The sensing matrix satisfies the coherence bound**

$$\mu(\Phi) < \bar{\mu} + \frac{C}{\sqrt{n}}$$

with high probability where

$$\bar{\mu} \le (1+\omega\ell)^{-1/2} c_\tau \|f\|_{\tau,\infty} \quad (d=2)$$
$$\bar{\mu} \le (1+\omega\ell)^{-1} c_1 \|f\|_{1,\infty} \quad (d=3)$$

where f is the sampling pdf over the unit circle/sphere and $\|\cdot\|_{\tau,\infty}$ is the Hölder norm of order $\tau > 1/2$ and the constant c_τ depends only on τ.

- **Proof uses concentration inequality and stationary phase.**
- **Do not need full view: supp$(f) \subset S^{d-1}, d = 2, 3$.**
- **Need some smoothness in f.**
- **To have $\mu \ll 1$, need $\omega\ell \gg 1$ and $n \gg 1$.**

Operator norm bound

THEOREM (F 09) **For the SIMO measurement we have**

$$\|\Phi\|^2 \le \frac{2N}{M}$$

with probability larger than

$$\left(1 - c_1\sqrt{\frac{M-1}{N}}\right)^{M(M-1)}$$

- Tropp 08, Candes-Plan 09:

coherence & operator norm bounds

→ # recoverable targets ~ # data

Recovery of potential

Foldy-Lax equation

$$u(\mathbf{r}_i) = u^{\mathsf{i}}(\mathbf{r}_i) + \omega^2 \sum_{j \neq i} G(\mathbf{r}_i, \mathbf{x}_j) V_j u(\mathbf{x}_j)$$

Define

$$\begin{aligned} U^{\mathsf{i}} &= (u^{\mathsf{i}}(\mathbf{r}_{i_1}), \ldots, u^{\mathsf{i}}(\mathbf{r}_{i_s}))^T \in \mathbb{C}^s \\ U &= (u(\mathbf{r}_{i_1}), \ldots, u(\mathbf{r}_{i_s}))^T \in \mathbb{C}^s \\ \mathbf{G} &= [(1 - \delta_{jl}) G(\mathbf{r}_{i_j}, \mathbf{r}_{i_l})] \\ \mathbf{V} &= \mathrm{diag}(V_{i_1}, \ldots, V_{i_s}). \end{aligned}$$

Foldy-Lax equation can be written as

$$\begin{aligned} U &= U^{\mathsf{i}} + \omega^2 \mathbf{G} \mathbf{V} U = U^{\mathsf{i}} + \omega^2 \mathbf{G} X' \\ X' &= (\textbf{nonzero components of } X) \end{aligned}$$

which implies

$$\mathbf{V} = \mathrm{diag}\left[\frac{X'}{\omega^2 \mathbf{G} X' + U^{\mathsf{i}}}\right]$$

THEOREM (F 09)

Suppose

$$\omega^{-2} \text{ is not an eigenvalue of the matrix } \mathbf{GV}$$

and

U^{i} **is not orthogonal to any row vector of** $\left(\mathbf{I} - \omega^2 \mathbf{GV}\right)^{-1}$

(to avoid zero divisor). Then

$$(*) \quad \mathbf{V} = \hat{\mathbf{V}}, \quad \hat{\mathbf{V}} = \mathrm{diag}\left[\frac{X'}{\omega^2 \mathbf{G} X' + U^{\mathrm{i}}}\right]$$

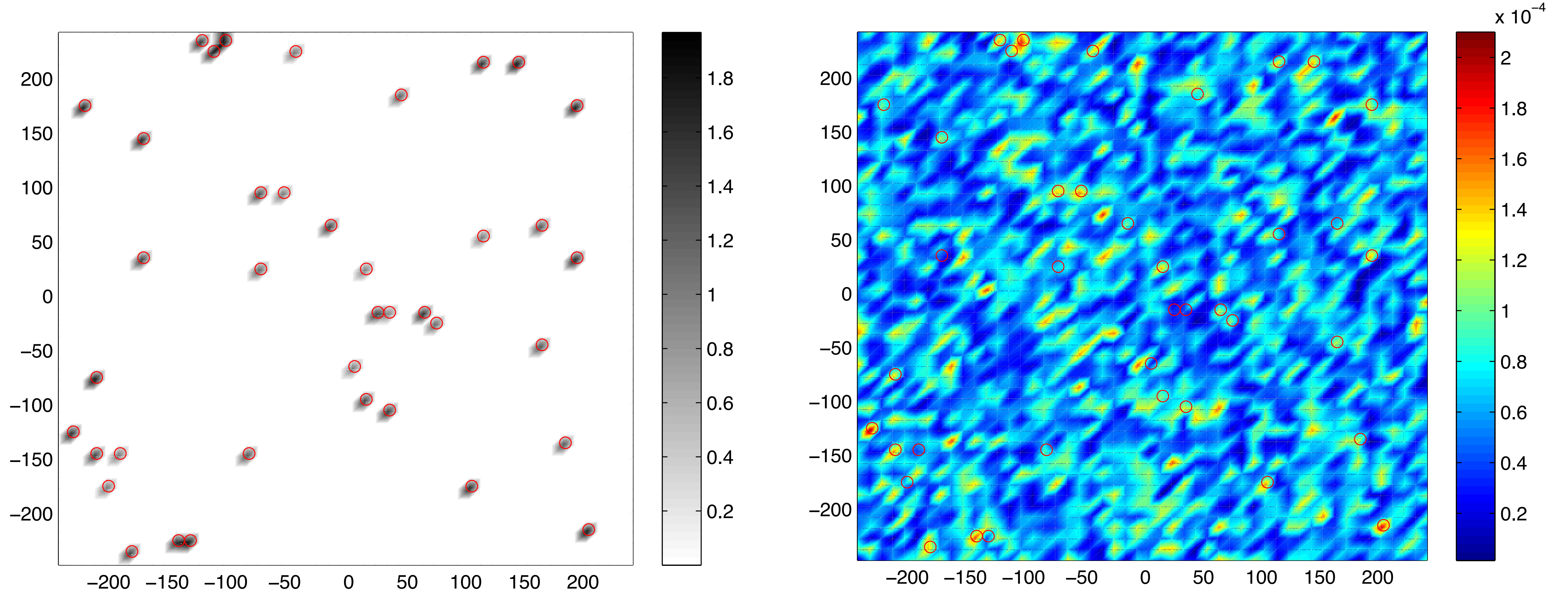
Let ε be the noise level in each measurement. If, in addition,

$$s\mu \leq 1/3, \quad \omega^2 \|\mathbf{GV}\| < 1/2 - \mathcal{O}(\varepsilon)$$

then $(*)$ is well defined and

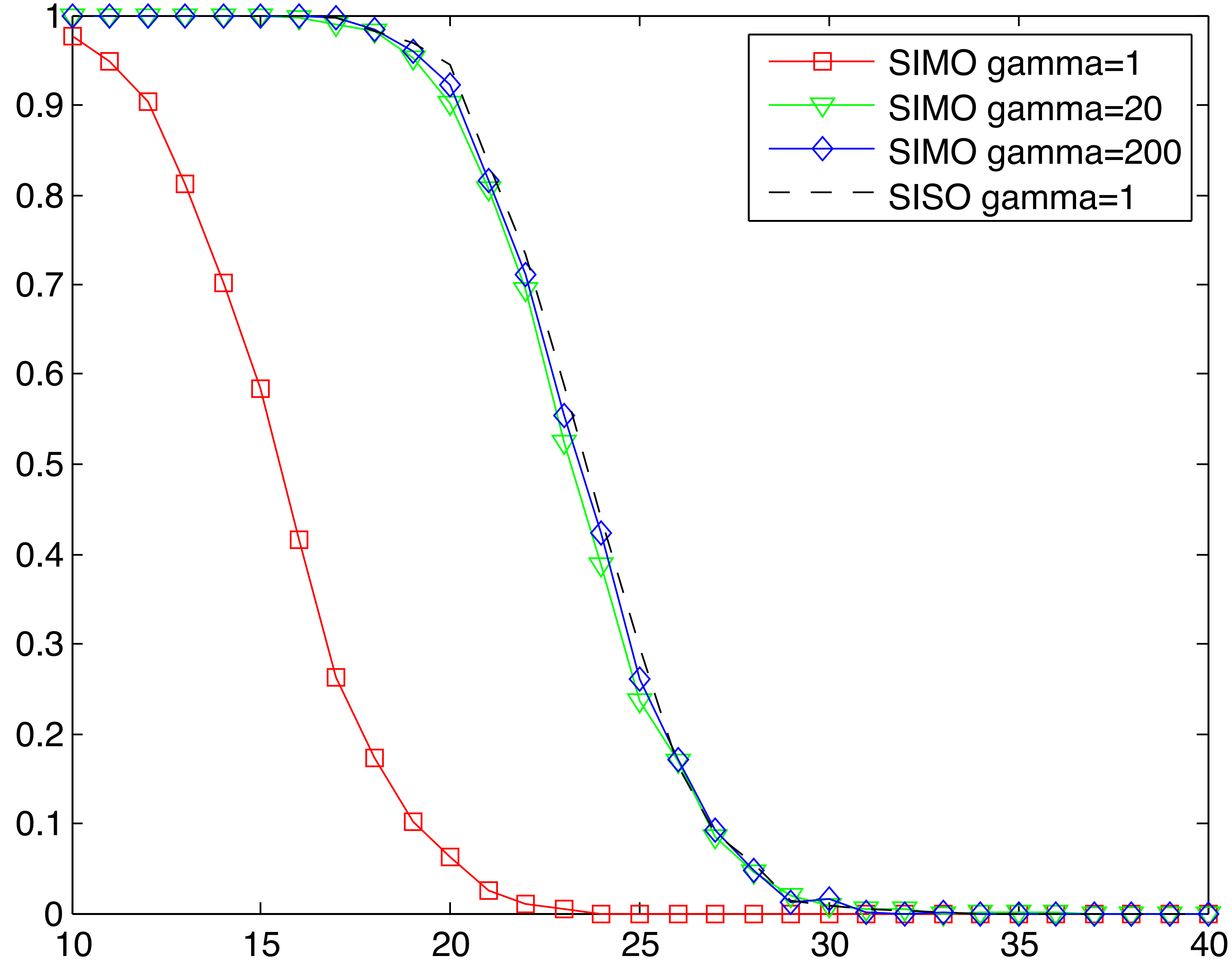
$$\|\mathbf{V} - \hat{\mathbf{V}}\| = \mathcal{O}(\varepsilon)$$

SIMO Reconstruction



40 objects, 121 data

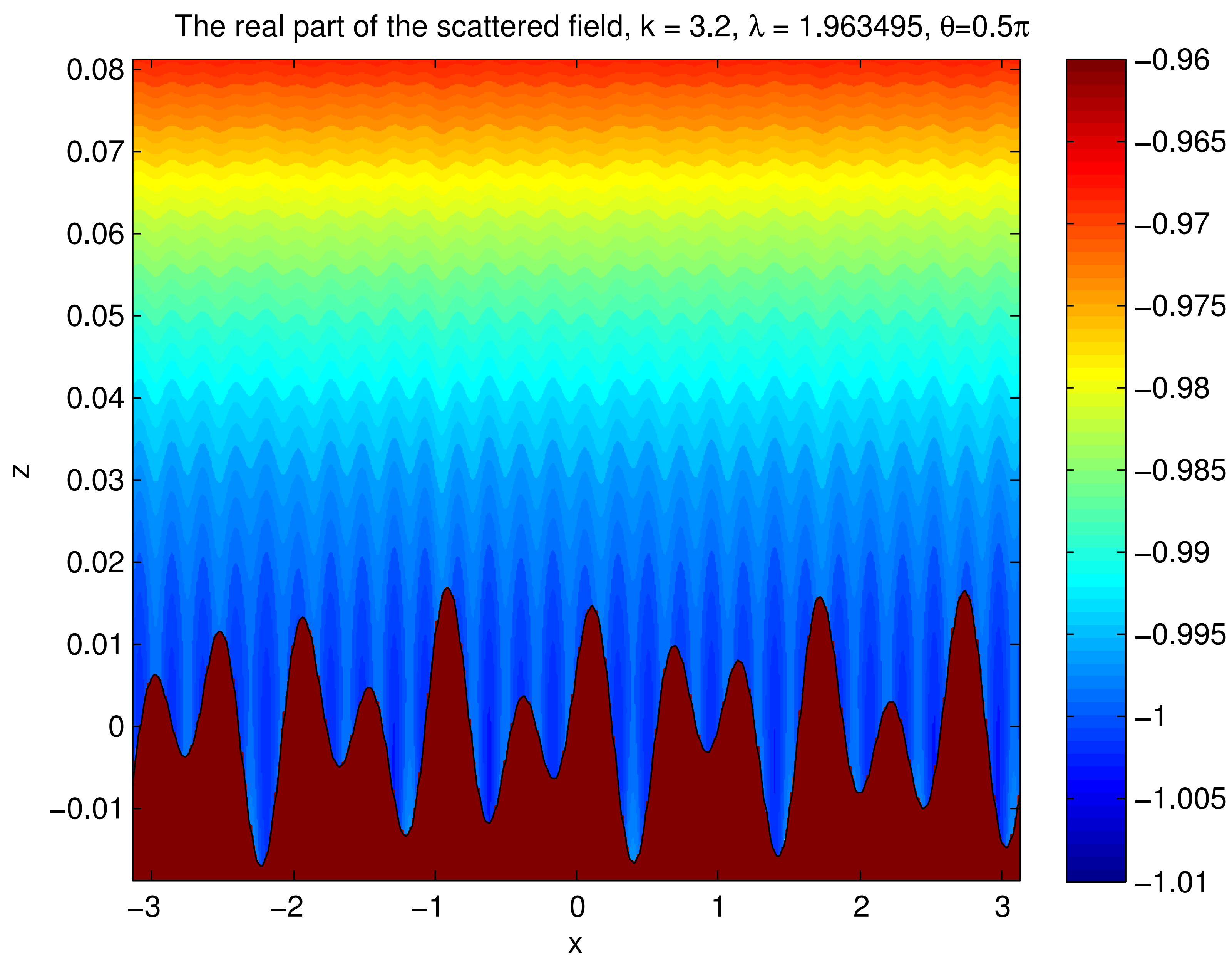
SIMO-SISO Comparison



Success probability versus # targets

Scattering by rough topography

With Hsiao-Chieh Tseng



Sound-soft BC

Angular spectrum

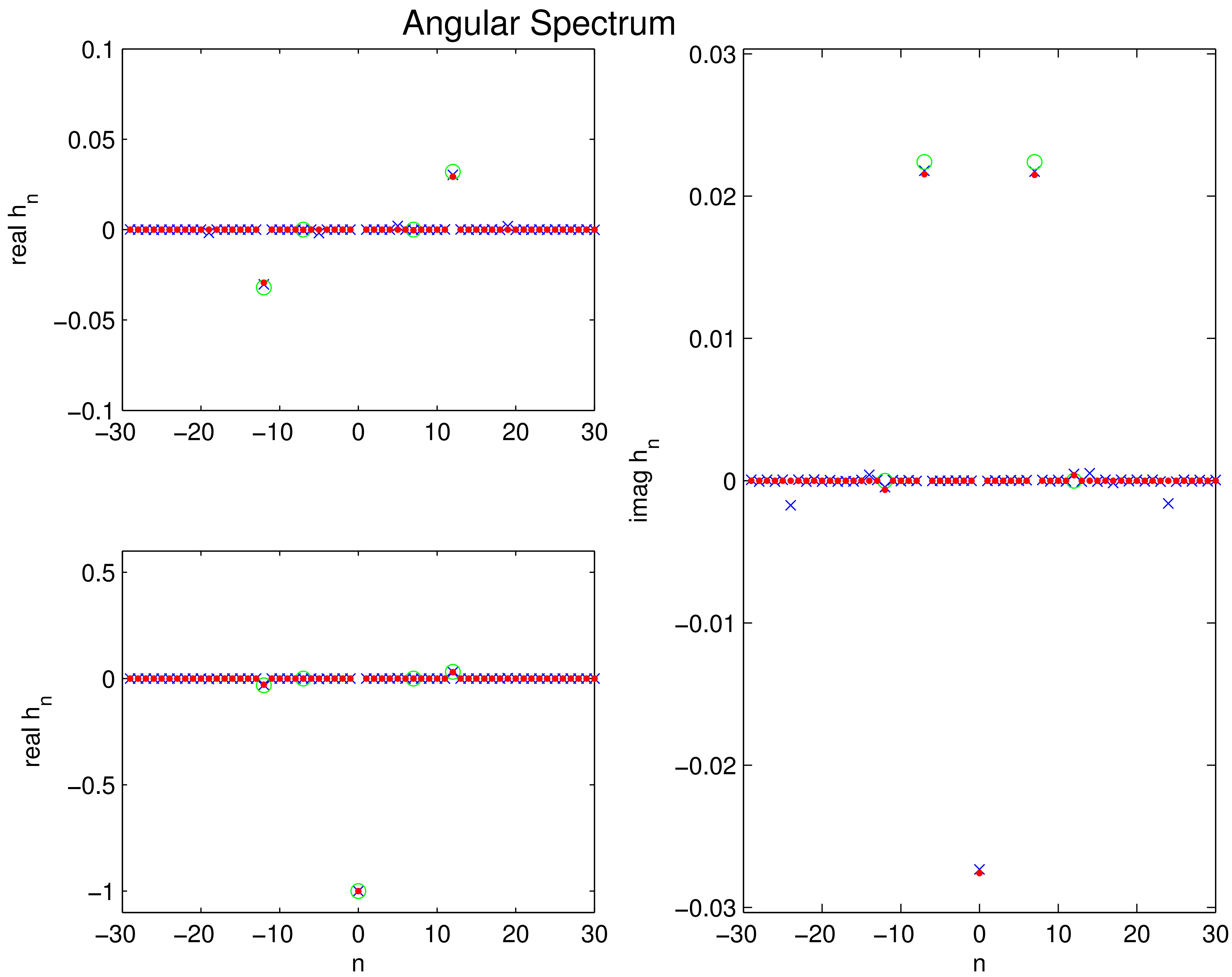
$$u^s(x,z) = \int_{-\infty}^{\infty} e^{\mathrm{i}\omega(\alpha x+\beta z)} u^s(\alpha) d\alpha, \qquad z > h_{\mathrm{max}} \text{ (the peak)}$$

$$u^s(\alpha) = \frac{-\mathrm{i}}{4\pi\beta} \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega(\alpha x+\beta h(x))} \frac{\partial u}{\partial \nu}\Big|_{z=h(x)} \sqrt{1+\dot{h}^2(x)}\, dx$$

$$\beta = \begin{cases} \sqrt{1-\alpha^2}, & |\alpha| \le 1 \quad \textbf{Outgoing mode} \\ \mathrm{i}\sqrt{\alpha^2-1}, & |\alpha| > 1. \quad \textbf{Evanescent mode} \end{cases}$$

- Data → angular spectrum
- Shallow groove → sparse AS

Sparsity of angular spectrum



Iterative solver

$$-u^{\mathsf{i}}(x,h(x)) \;=\; \frac{1}{2}\psi(x)+\int\Bigg(\frac{\partial}{\partial\nu'}G\Big((x,h(x)),(x',h(x'))\Big) \\ -\mathsf{i}\eta G\Big((x,h(x)),(x',h(x'))\Big)\Bigg)\psi(x')\sqrt{1+\dot{h}^2(x')}dx'$$

$$u^s(\alpha) \;=\; \frac{1}{4\pi}\int \mathsf{e}^{-\mathsf{i}\omega\left(\alpha x'+\beta h(x')\right)}\psi(x') \\ \times\left(\omega-\omega\dot{h}(x')\frac{\alpha}{\beta}+\frac{\eta}{\beta}\sqrt{1+\dot{h}^2(x')}\right)dx'$$

- Frechet differentiable ⟹ Newton iteration

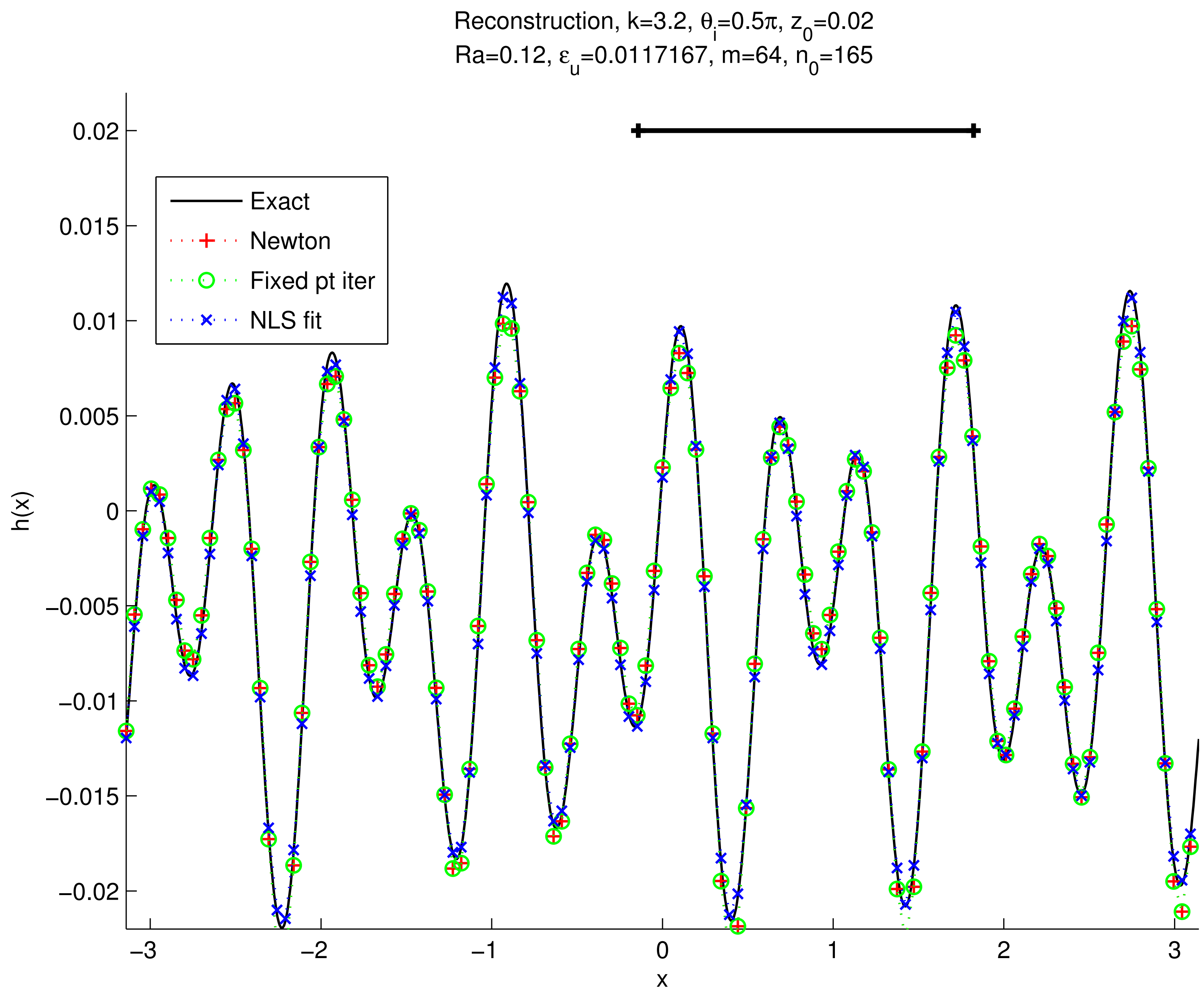
Rayleigh hypothesis

$$u^s(x,z) \quad = \quad \int e^{\mathrm{i}\omega(\alpha x+\beta z)} u^s(\alpha) d\alpha, \qquad \text{in the groove}$$

$$-u^i(x,h(x)) \quad \overset{\text{BC}}{=} \quad u^s(x,h(x)) = \int e^{\mathrm{i}\omega(\alpha x+\beta h(x))} u^s(\alpha) d\alpha$$

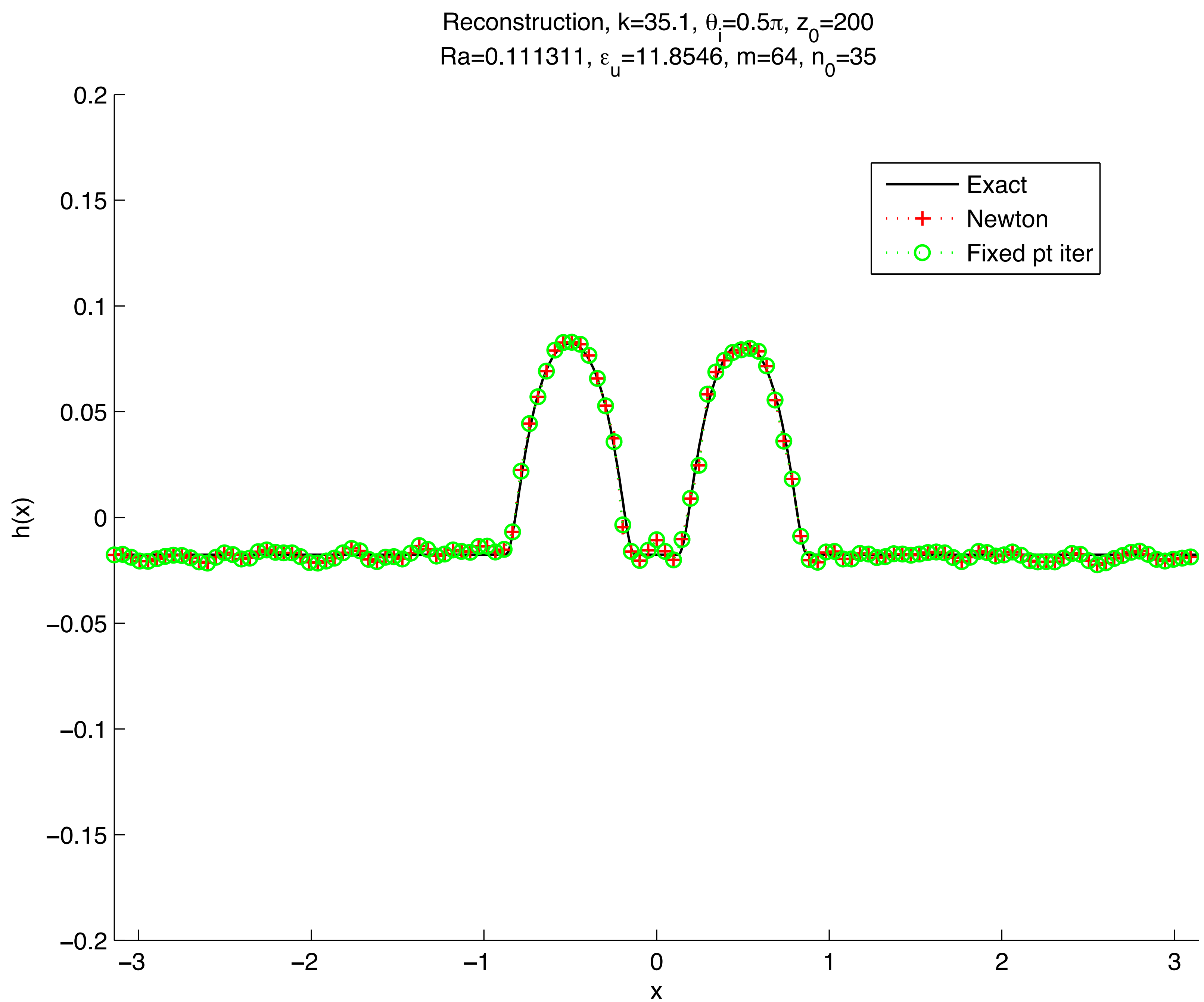
- Valid for small, but finite roughness
- Point-wise recovery: Newton method
- Nonlinear least squares with sparse Fourier modes

Near-field Reconstruction



data = 64, 2-mode profile

Far-field reconstruction



data = 64

Phase retrieval

Discrete finite objects

Let $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$ and $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$.

multi-index : $\quad \mathbf{z}^{\mathbf{n}} = z_1^{n_1} z_2^{n_2}$

Let the object be represented by $f(\mathbf{n}), \mathbf{n} \leq \mathbf{N} = (N, N)$

Fourier transform describes wave propagation

$$F(e^{i2\pi w_1}, e^{i2\pi w_2}) = \sum_{\mathbf{n}} f(\mathbf{n}) e^{-i2\pi \mathbf{n}\cdot\mathbf{w}}$$

Analytic continuation $\Longrightarrow$ z-transform

$$F(\mathbf{z}) = \sum_{\mathbf{n}} f(\mathbf{n}) \mathbf{z}^{-\mathbf{n}}.$$

Discrete phase retrieval problem:

Determine $f(\mathbf{n})$ from Fourier magnitude data

$$|F(\mathbf{w})|, \quad \forall \mathbf{w} = (e^{i2\pi w_1}, e^{i2\pi w_2}) \in [0,1]^2$$

Fourier magnitude data:

$$|F(\mathbf{w})|^2 = \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \sum_{\mathbf{m}} f(\mathbf{m}+\mathbf{n}) f^*(\mathbf{m}) e^{-i2\pi\mathbf{n}\cdot\mathbf{w}}$$
$$= \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \mathcal{C}_f(\mathbf{n}) e^{-i2\pi\mathbf{n}\cdot\mathbf{w}}$$

where

$$\mathcal{C}_f(\mathbf{n}) = \sum_{\mathbf{m}} f(\mathbf{m}+\mathbf{n}) f^*(\mathbf{m})$$

is the autocorrelation function of f.

Fourier magnitude data contain complete information about autocorrelation function.

Sampling Theorem:
supp$(C_f) \subset [-N, N]^2 \Longrightarrow [0,1]^2$ **is reduced to the Nyquist grid**

$$\mathcal{M} = \left\{(k_1, k_2) : k_j = 0, \frac{1}{2N+1}, \frac{2}{2N+1}, \cdots, \frac{2N}{2N+1}\right\}$$

Trivial ambiguities

Autocorrelation:

$$\mathcal{C}_f(\mathbf{n}) = \sum_{\mathbf{m}+\mathbf{n}\in\mathcal{N}} f(\mathbf{m}+\mathbf{n}) f^*(\mathbf{m})$$

Invariant under:

(i) global phase,

$$f(\mathbf{n}) \longrightarrow e^{\mathrm{i}\theta} f(\mathbf{n}), \quad \textbf{for some } \theta \in [0, 2\pi],$$

(ii) spatial translation

$$f(\mathbf{n}) \longrightarrow f(\mathbf{n}+\mathbf{m}), \quad \textbf{some } \mathbf{m} \in \mathbb{Z}^2$$

(iii) conjugate inversion (twin image)

$$f(\mathbf{n}) \longrightarrow f^*(\mathbf{N}-\mathbf{n}).$$

Nontrivial ambiguity

THEOREM (Hayes 82, Pitts-Greenleaf 03)

Let the z-transform $F(\mathbf{z})$ of a finite complex-valued sequence $\{f(\mathbf{n})\}$ be given by

$$F(\mathbf{z}) = \alpha \mathbf{z}^{-\mathbf{m}} \prod_{k=1}^{p} F_k(\mathbf{z}), \qquad \mathbf{m} \in \mathbb{N}^2, \alpha \in \mathbb{C}$$

where $F_k, k = 1, ..., p$ are nontrivial irreducible polynomials. Let $G(\mathbf{z})$ be the $\mathbf{z}$-transform of another finite sequence $g(\mathbf{n})$. Suppose $|F(\mathbf{w})| = |G(\mathbf{w})|, \forall \mathbf{w} \in [0, 1]^2$. Then $G(\mathbf{z})$ must have the form

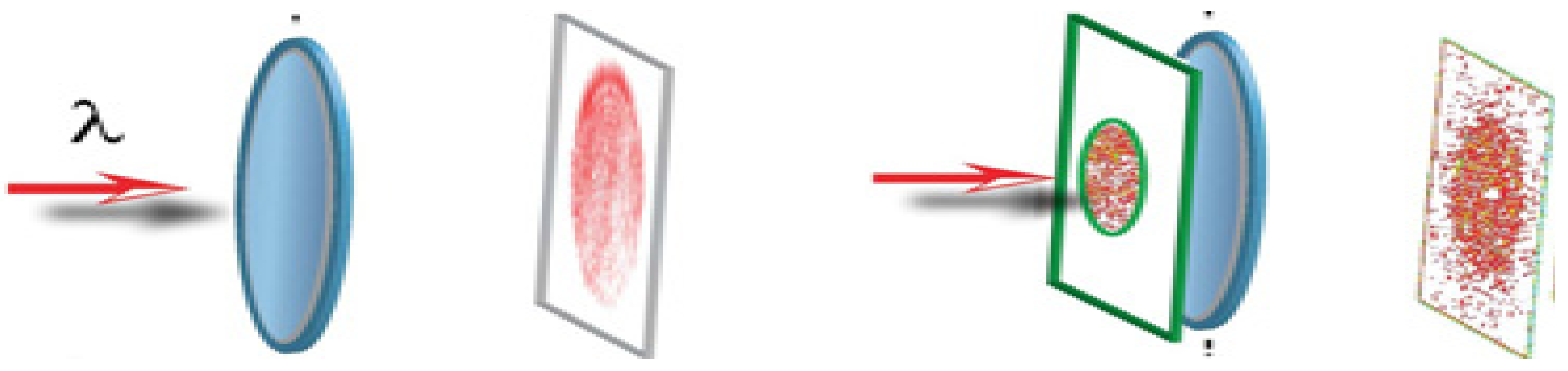
$$G(\mathbf{z}) = |\alpha| e^{\mathrm{i}\theta} \mathbf{z}^{-\mathbf{p}} \left(\prod_{k \in I} F_k(\mathbf{z}) \right) \left(\prod_{k \in I^c} F_k^*(1/\mathbf{z}^*) \right), \qquad \mathbf{p} \in \mathbb{N}^2, \theta \in \mathbb{R}$$

where I is a subset of $\{1, 2, ..., p\}$.

Nontrivial ambiguity: Partial conjugate inversion on factors.

Random illumination

Coded aperture imaging



Diffuser generated speckle pattern: Garcia-Zalevsky-Fixler 05

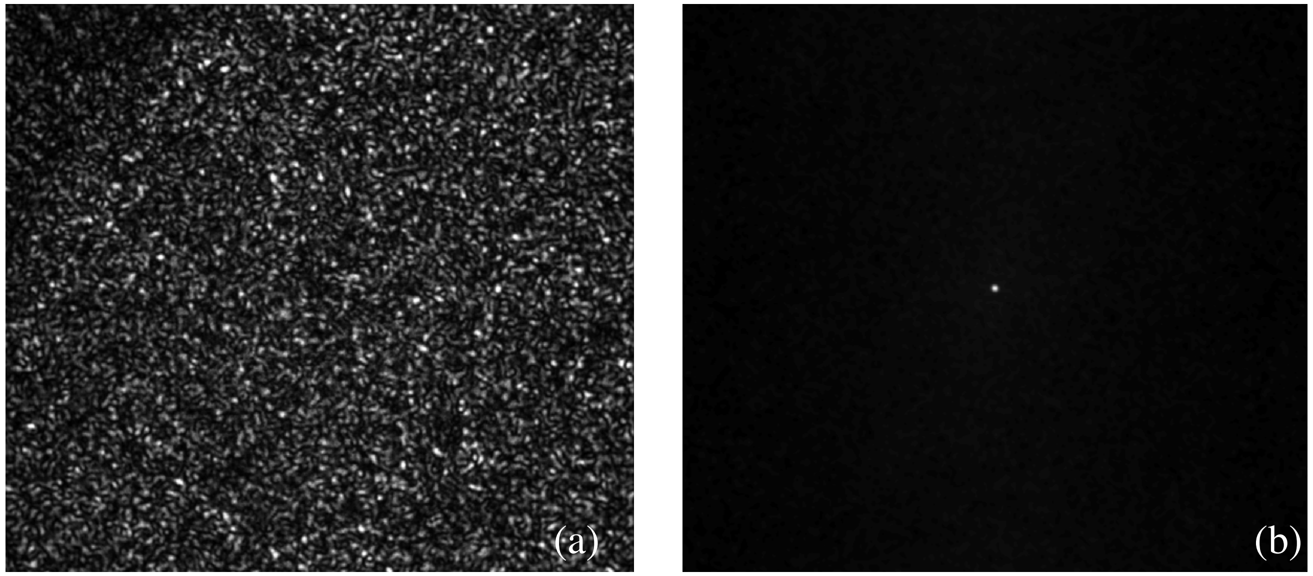


Fig. 2. (a) Encoding speckle pattern. (b) Autocorrelation of the encoding pattern.

Random illumination

$$\tilde{f}(\mathbf{n}) = f(\mathbf{n})\lambda(\mathbf{n}) \quad \text{(illuminated object)}$$

$\lambda(\mathbf{n})$, representing the illumination field, is a known sequence of samples of random variables.

Let $\lambda(\mathbf{n})$ be continuous random variables with respect to the Lebesque measure on $\mathbb{S}^1$ (the unit circle), $\mathbb{R}$ or $\mathbb{C}$.

Case of $\mathbb{S}^1$ can be facilitated by a random phase modulator with

$$\lambda(\mathbf{n}) = e^{\mathrm{i}\phi(\mathbf{n})}$$

where $\phi(\mathbf{n})$ are continuous random variables on $[0, 2\pi]$.
Case of $\mathbb{R}$: random amplitude modulator.
Case of $\mathbb{C}$: both phase and amplitude modulations.

Irreducibility

THEOREM. Suppose that the support of the object $\{f(\mathrm{n})\}$ has rank ≥ 2. Then the the z-transform of the illuminated object $f(\mathrm{n})\lambda(\mathrm{n})$ is irreducible with probability one.

False for rank 1 objects: fundamental thm of algebra

$$\sigma = \frac{\text{Fourier magnitude data number}}{\text{unknown image pixel number}}$$

$= \quad 2^d$ Oversampling ratio

We assume object rank > 1 below

Absolute uniqueness

Positivity

THEOREM If $f(\mathrm{n})$ is real and nonnegative for every n then, with probability one, f is determined absolutely uniquely by the Fourier magnitude measurement on the lattice $\mathcal{L}$.

Sector constraint

THEOREM Suppose the phases of the object belong to $[a, b] \subset [0, 2\pi]$. Then the solution to the Fourier phasing problem has a unique solution with probability exponentially close to unity (depending on the sparsity and the phase range $|b - a|$.)

Complex objects w/o constraint

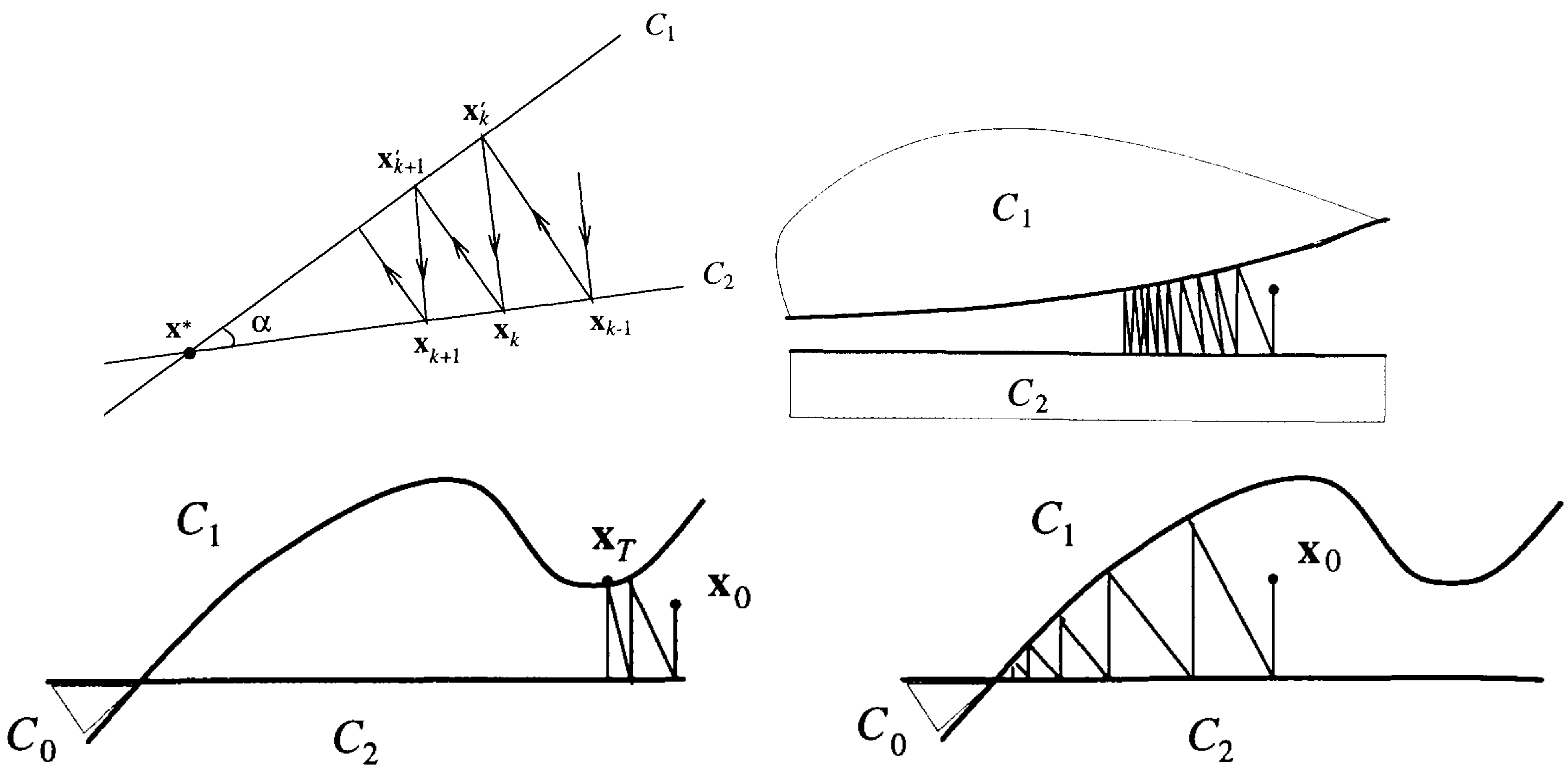
THEOREM. Suppose that $\{\lambda_1(n)\}$ are i.i.d. continuous random variables with respect to the Lebesgue measure on $\mathbb{S}^1$, $\mathbb{R}$ or $\mathbb{C}$ and in addition either one of the following conditions is true.

(i) $\{\lambda_2(n)\}$ are i.i.d. continuous random variables with respect to the Lebesgue measure on $\mathbb{S}^1$, $\mathbb{R}$ or $\mathbb{C}$ and $\{\lambda_2(n)\}$ are independent of $\{\lambda_1(n)\}$.

(ii) $\{\lambda_2(n)\}$ are deterministic.

Then with probability one $f(n)$ is uniquely determined, up to a constant phase factor, by the Fourier magnitude measurements with two illuminations λ_1 and λ_2.

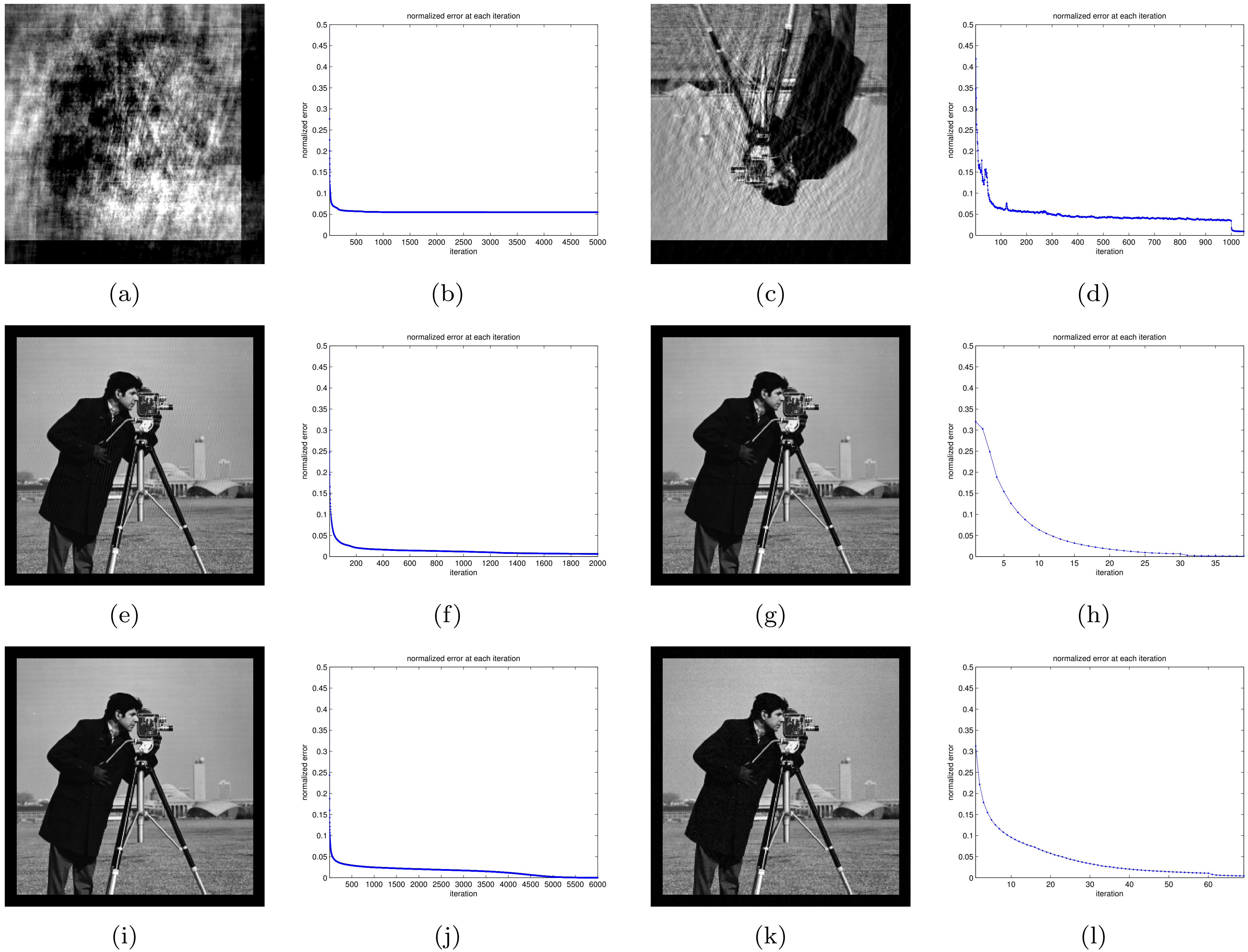
Error Reduction (Gerchberg-Saxton)



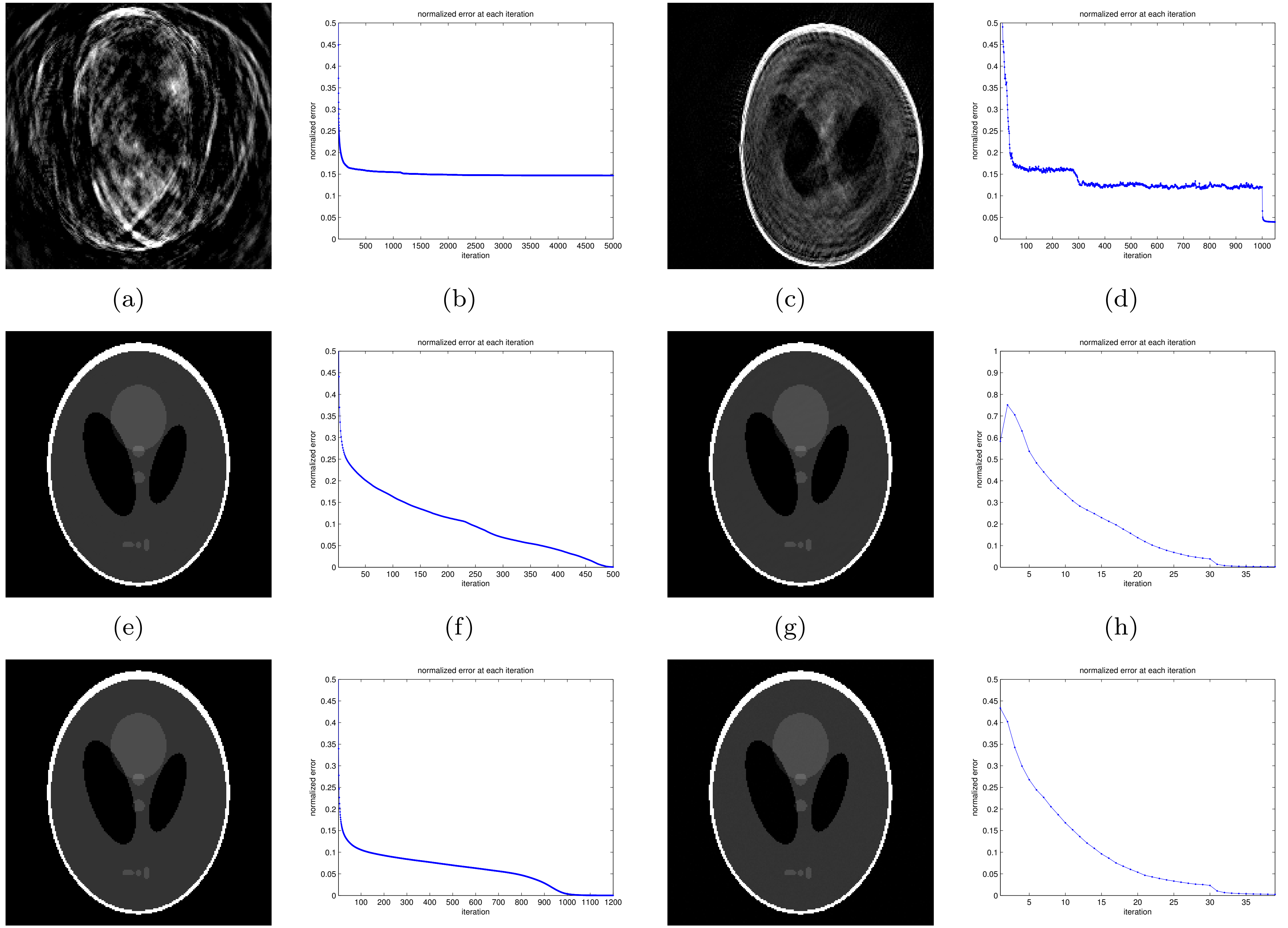
Bregman 65: convex constraints $\Longrightarrow$ convergence to a feasible solution.

Fourier magnitude data are a non-convex constraint!

Nonconvexity or nonuniqueness ?



(e)-(h) Low resolution 40 x 40 block illumination with OR=2
(i)-(l) High resolution illumination with OR=1

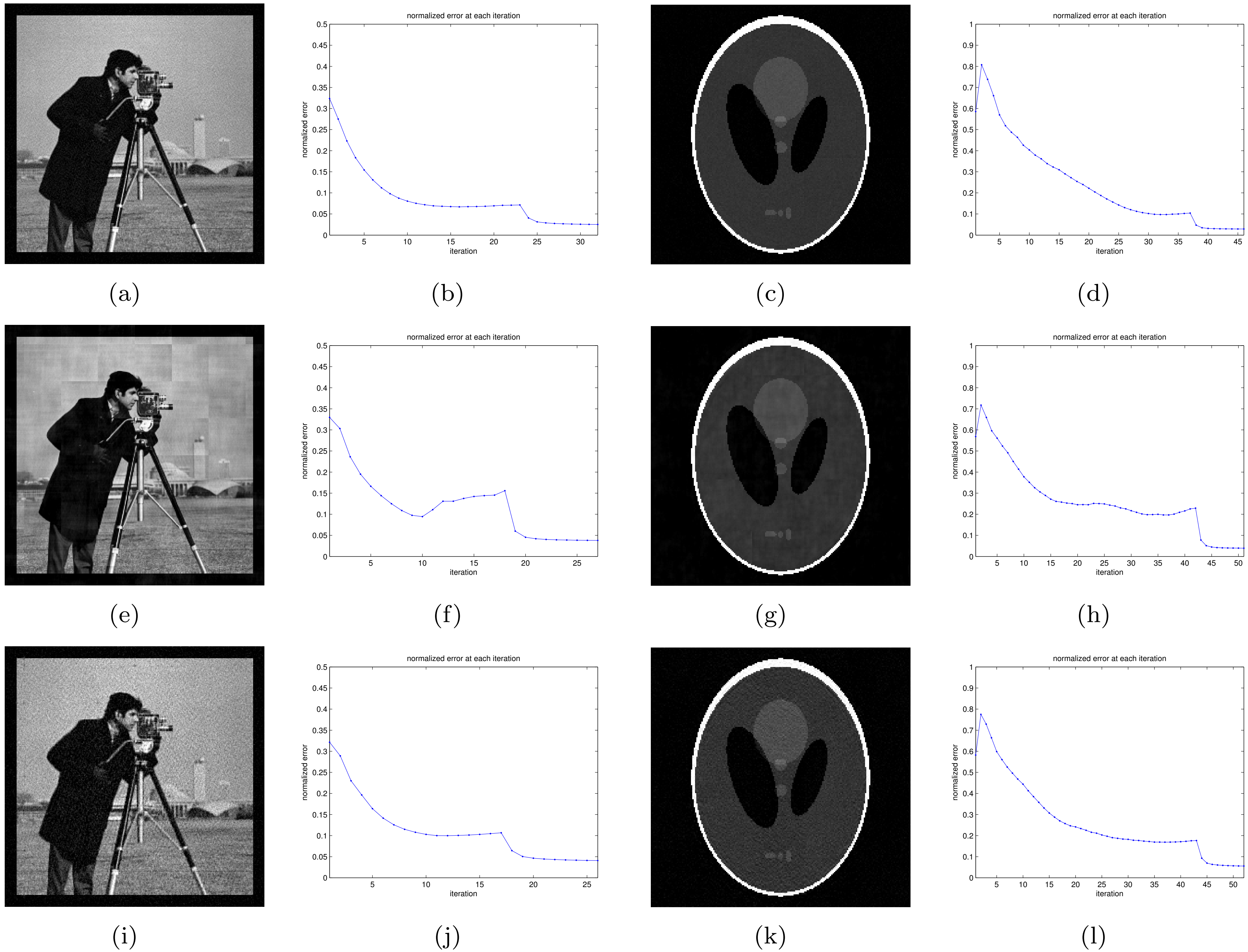


(a) (b) (c) (d)

(e) (f) (g) (h)

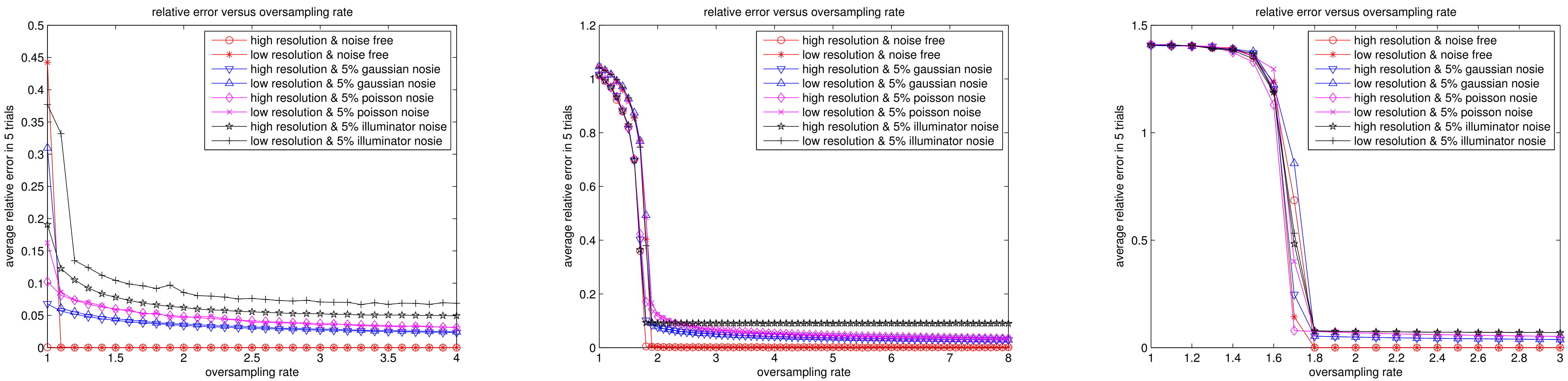
(i) (j) (k) (l)

(e) - (h) Low resolution 40 x 40 block illumination with OR=2
(i) - (l) High resolution illumination with OR=1



Low resolution illumination with 5% Gaussian, Poisson and illuminator errors

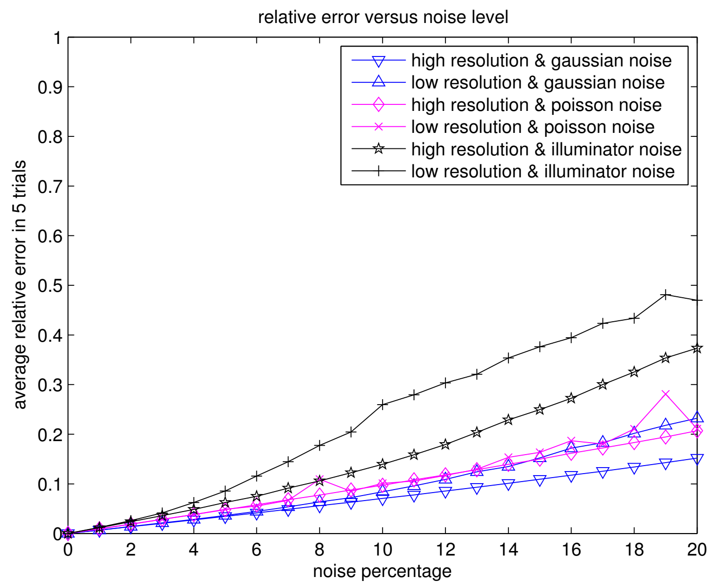
Compressed measurement



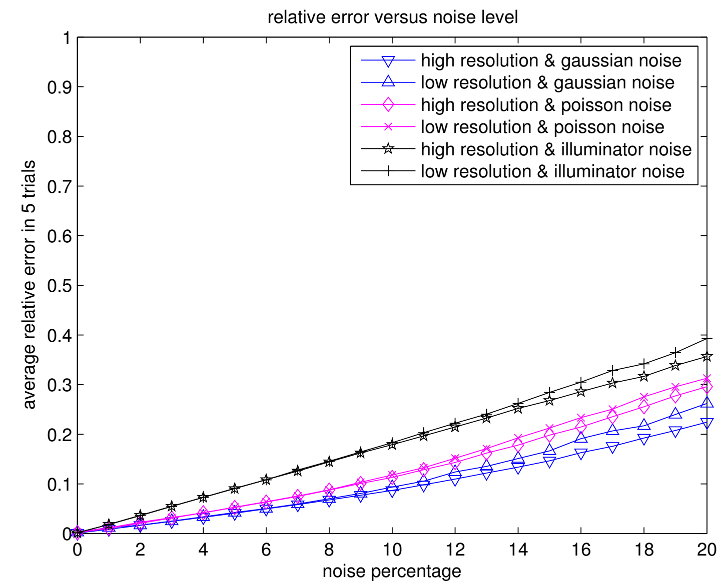
(a) (b) (c)

(a) real-valued
(b) positive real & imaginary parts
(c) no constraint

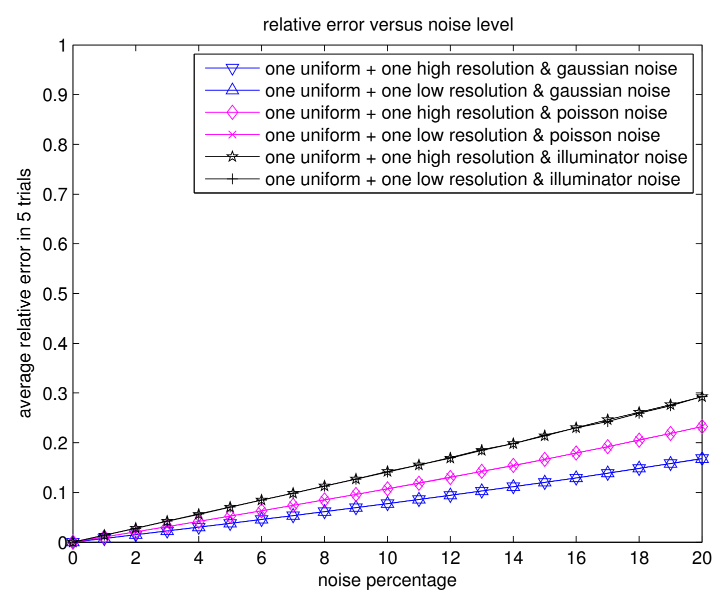
Noise stability



(a)



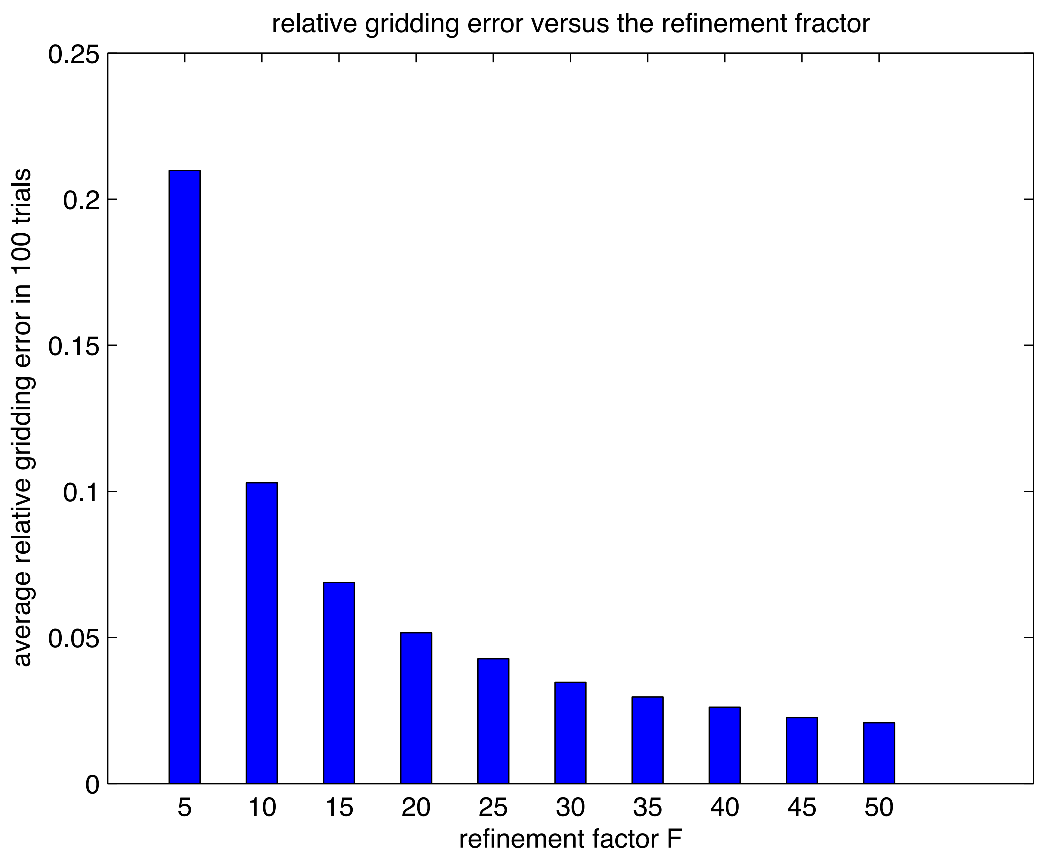
(b)

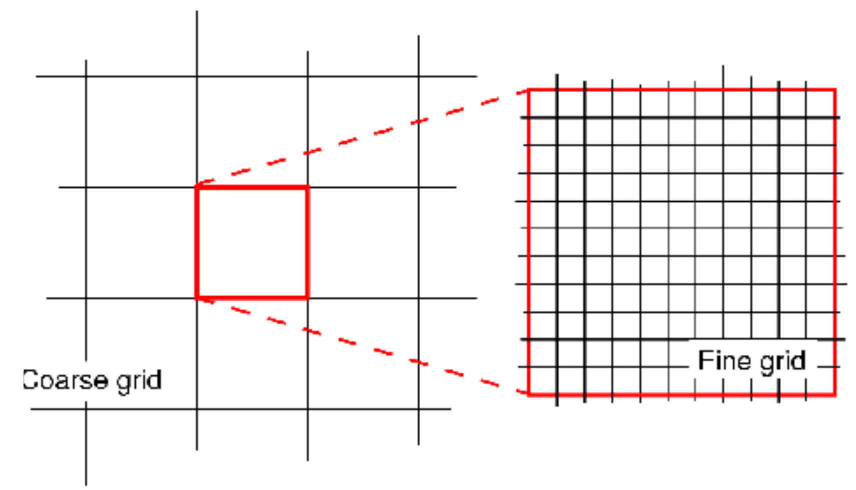


(c)

(a) real-valued objects
(b) positive real & imaginary parts
(c) no constraint

Highly coherent matrix

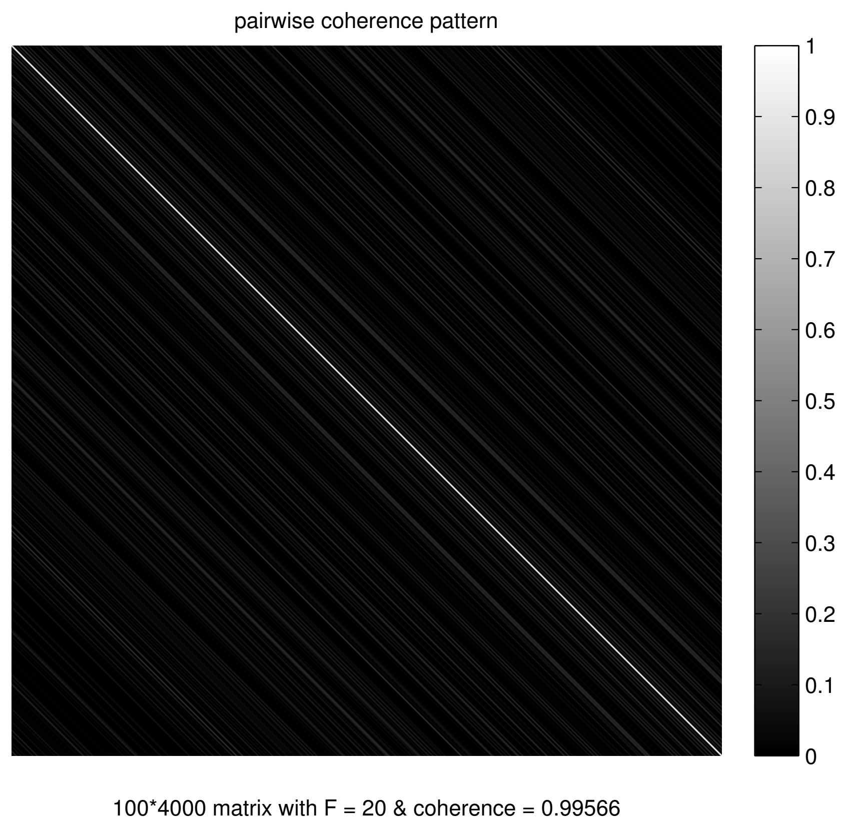


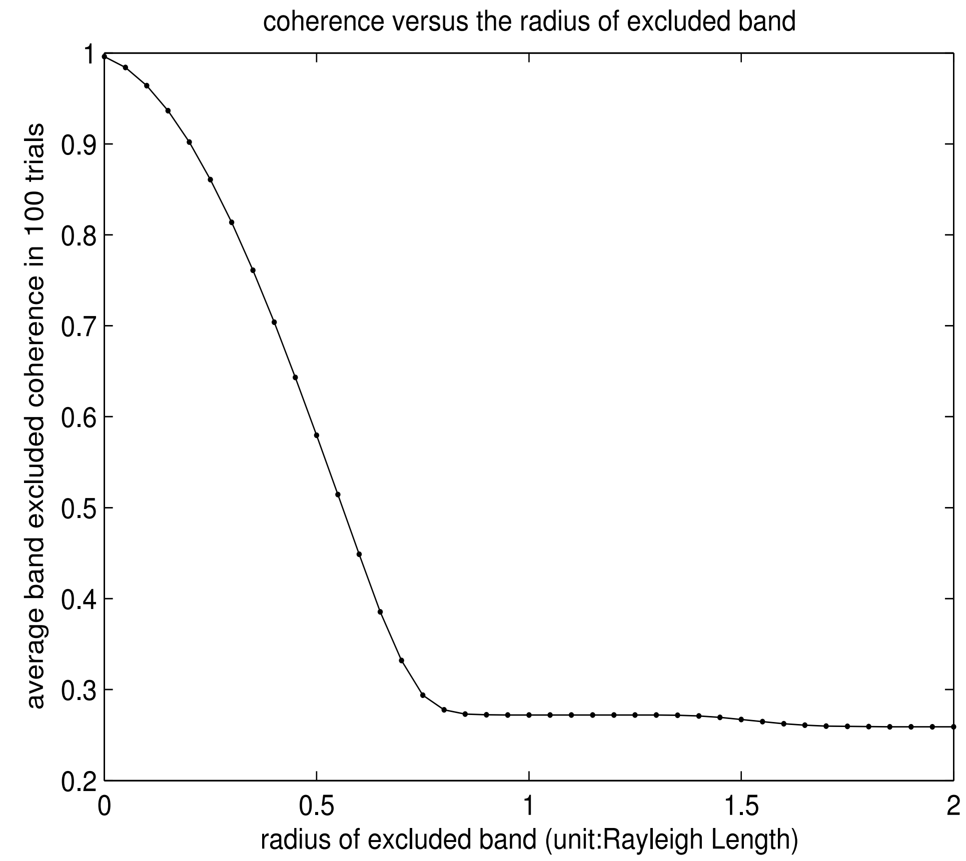


Gridding error is inversely proportional to refinement factor F

$$\mathcal{G} = \mathbb{Z}/F$$

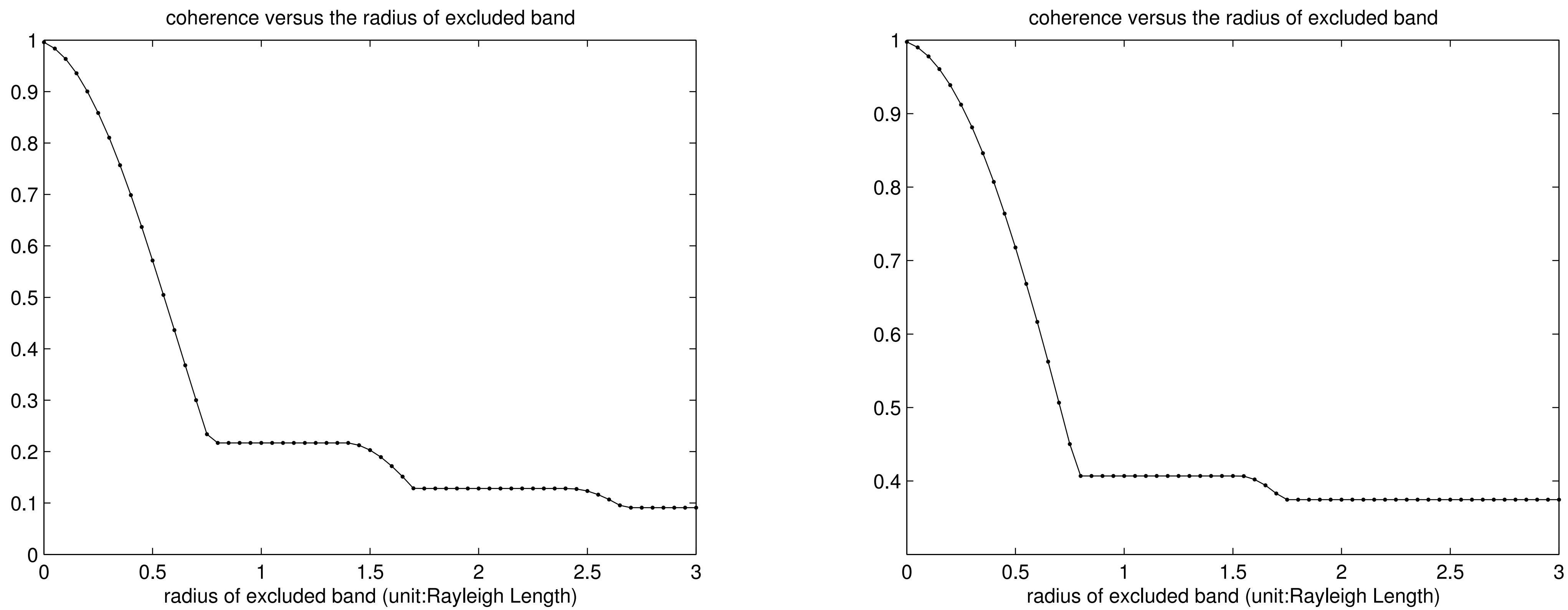
Coherence band





Coherence pattern $\Phi^*\Phi$ for 100×4000 matrix with $F = 20$ (left)

Redundant dictionary



Coherence bands of the DFT frame $\mathbf{D}$ (left) and $\mathbf{\Phi} = \mathbf{AD}$ (right).

Coherence band

Let $\eta > 0$. Define the η-coherence band of the index k to be the set

$$B_\eta(k) = \{i \mid \mu(i,k) > \eta\},$$

and the η-coherence band of the index set S to be the set

$$B_\eta(S) = \cup_{k \in S} B_\eta(k).$$

Due to the symmetry $\mu(i,k) = \mu(k,i)$, $i \in B_\eta(k)$ if and only if $k \in B_\eta(i)$.

Denote

$$\begin{aligned} B_\eta^{(2)}(k) &\equiv B_\eta(B_\eta(k)) = \cup_{j \in B_\eta(k)} B_\eta(j) \\ B_\eta^{(2)}(S) &\equiv B_\eta(B_\eta(S)) = \cup_{k \in S} B_\eta^{(2)}(k). \end{aligned}$$

Band exclusion

Algorithm 1. BOMP

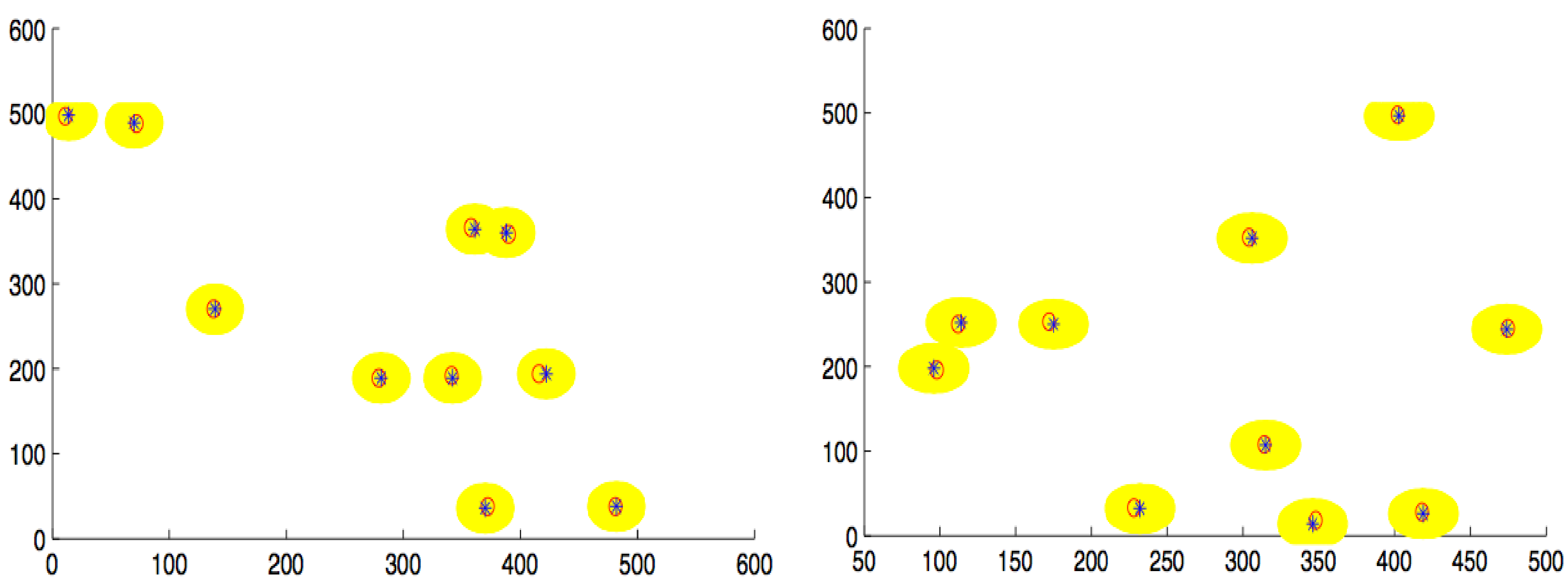
Input: $\mathbf{\Phi}, Y, \eta > 0$

Initialization: $X^0 = 0, R^0 = Y$ and $S^0 = \emptyset$

Iteration: For $n = 1, ..., s$

1) $i_{\text{max}} = \arg\max_i |\langle R^{n-1}, \Phi_i \rangle|, i \notin B_\eta^{(2)}(S^{n-1})$
2) $S^n = S^{n-1} \cup \{i_{\text{max}}\}$
3) $X^n = \arg\min_Z \|\mathbf{\Phi} Z - Y\|_2$ s.t. $\text{supp}(Z) \in S^n$
4) $R^n = Y - \mathbf{\Phi} X^n$

Output: $\mathbf{x}^s$.



BOMP: performance guarantee

THEOREM (F & Liao 11) **Suppose that**

$$B_\eta(i) \cap B_\eta^{(2)}(j) = \emptyset, \quad \forall i, j \in \textbf{supp}(X)$$

and that

$$(5s-4) \cdot \eta \cdot \frac{x_{\text{max}}}{x_{\text{min}}} + \frac{5}{2} \cdot \frac{\|E\|_2}{x_{\text{min}}} < 1$$

where

$$x_{\text{max}} = \max_k |X_k|, \quad x_{\text{min}} = \min_k |X_k|.$$

Let $\hat{X}$ **be the BOMP reconstruction. Then**

$$\textbf{supp}(\hat{X}) \subseteq B_\eta(\textbf{supp}(X))$$

and every nonzero component of $\hat{X}$ **is in the** η**-coherence band of a unique nonzero component of** X**.**

Theoretical resolution 3ℓ. Numerical resolution $\sim 1\ell$.

Independent of grid refinement!

Compression: $s \sim \frac{\sqrt{M}}{\text{dynamic range}}$, $\quad \eta \sim \frac{1}{\sqrt{M}}$ for moderate SNR.

Local optimization

Algorithm 2. Local Optimization (LO)

Input: $\mathbf{\Phi}, Y, \eta > 0, S^0 = \{i_1, \ldots, i_k\}$.
Iteration: For $n = 1, 2, ..., k$.
1) $X^n = \arg\ \min_Z \|\mathbf{\Phi} Z - Y\|_2, \quad \text{supp}(\mathbf{z}) = S^{n-1} \cup B_\eta(i_n).$
2) $S^n = \text{supp}(X^n).$
Output: S^k.

Algorithm 3. BLOOMP

Input: $\mathbf{\Phi}, Y, \eta > 0$
Initialization: $X^0 = 0, R^0 = Y$ and $S^0 = \emptyset$
Iteration: For $n = 1, ..., s$
1) $i_{\max} = \arg\min_i |\left\langle R^{n-1}, \Phi_i \right\rangle|, i \notin B_\eta^{(2)}(S^{n-1})$
2) $S^n = \text{LO}(S^{n-1} \cup \{i_{\max}\}).$
3) $X^n = \arg\min_Z \|\mathbf{\Phi} Z - Y\|_2$ s.t. $\text{supp}(Z) \in S^n$
4) $R^n = Y - \mathbf{\Phi} X^n$
Output: X^s.

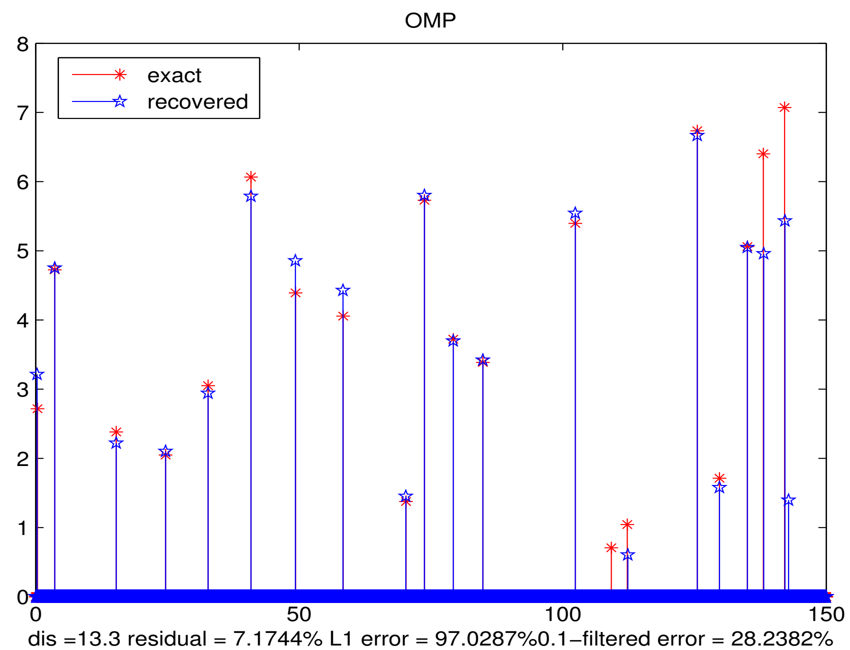
Performance guarantee

THEOREM (F & Liao 11) **Let S^0 and S^k be the input and output, respectively, of the LO algorithm.**

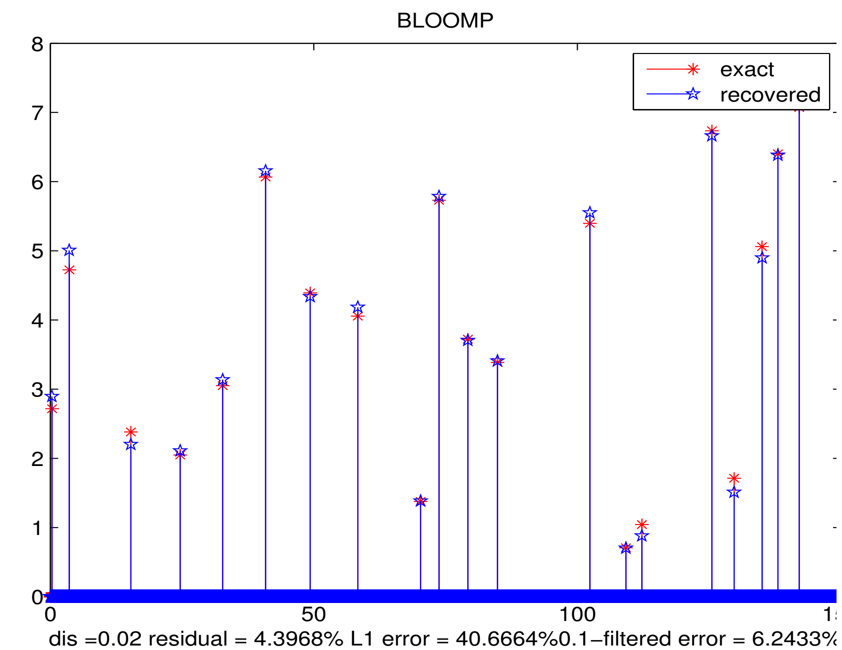
If

$$x_{\text{min}} > (\varepsilon + 2(s-1)\eta)\left(\frac{1}{1-\eta} + \sqrt{\frac{1}{(1-\eta)^2} + \frac{1}{1-\eta^2}}\right)$$

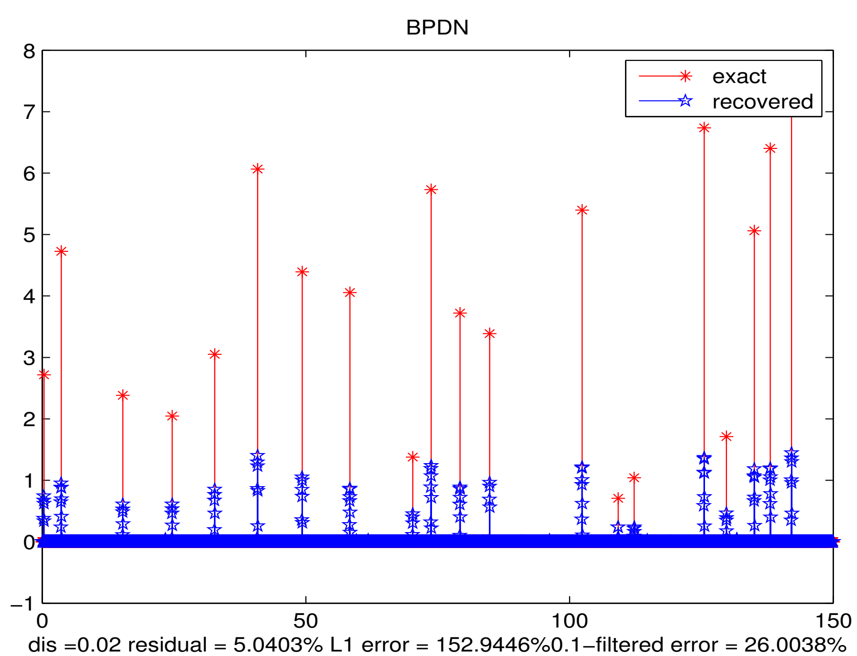
and each element of S^0 is in the η-coherence band of a unique nonzero component of X, then each element of S^k remains in the η-coherence band of a unique nonzero component of X.



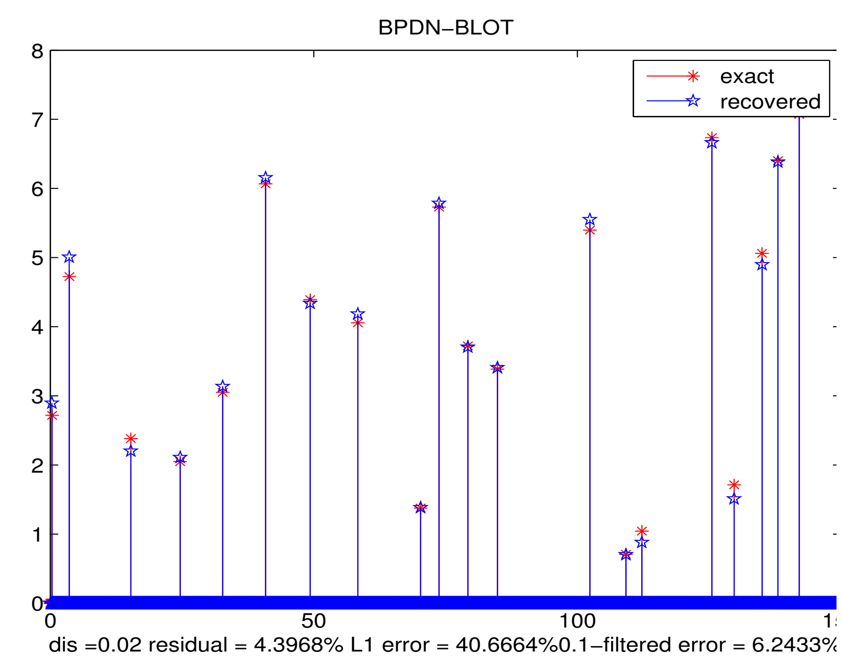
(a)



(b)

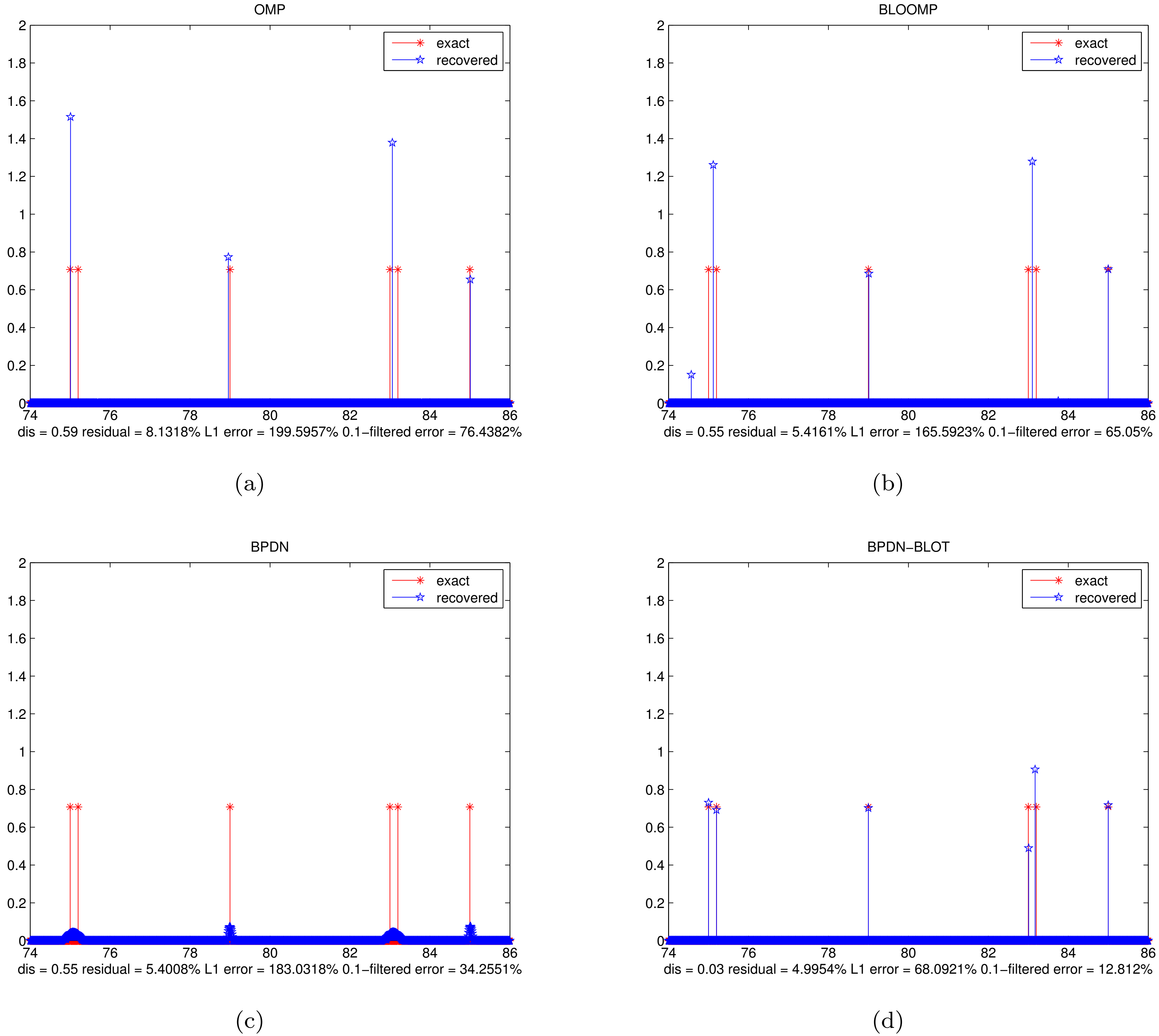


(c)



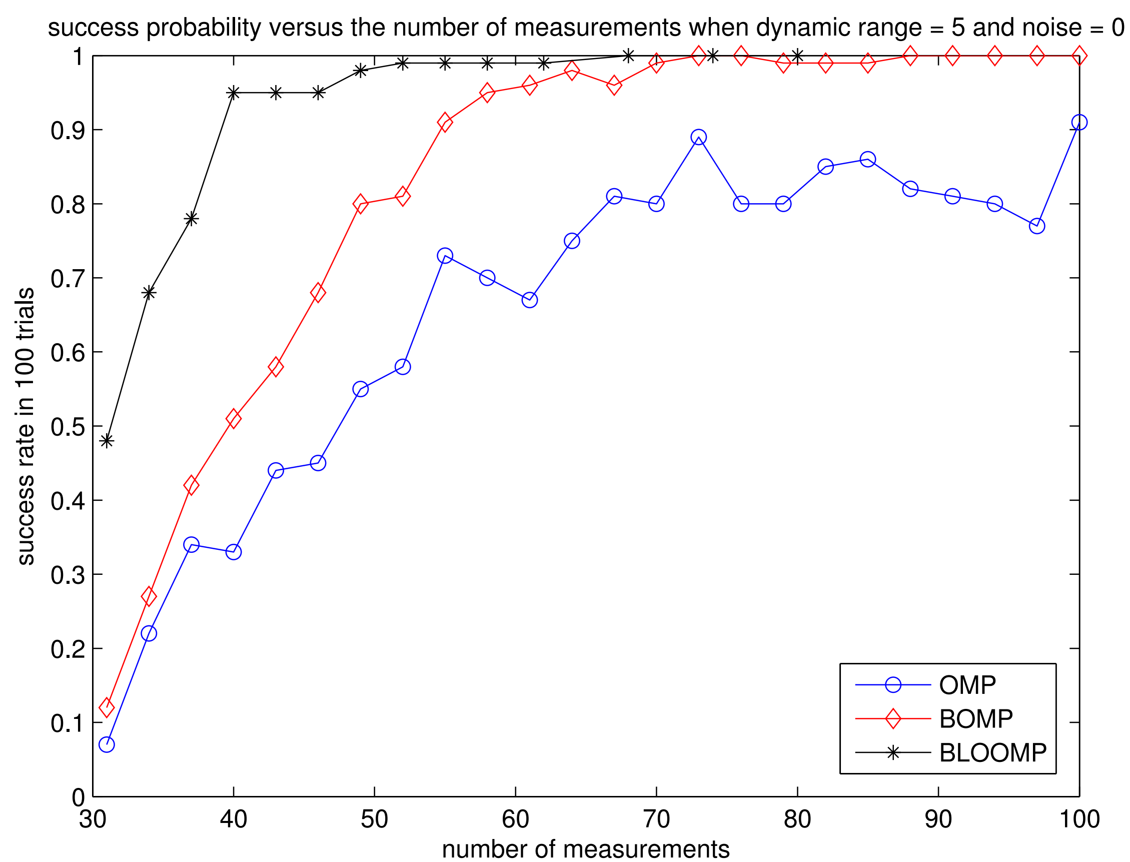
(d)

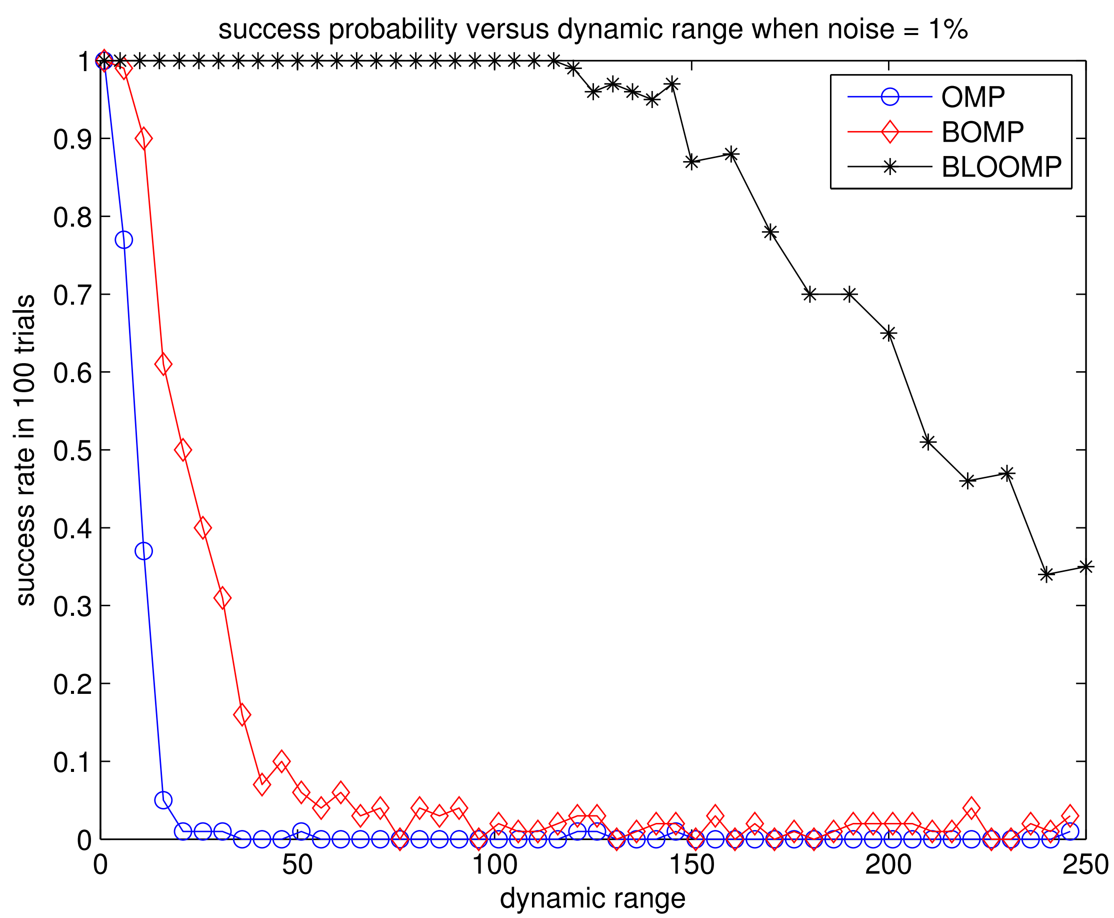
Reconstructions of real parts of 20 well-separated targets (distance > 3 Rayleigh length) with unresolved grid(superresolution factor = 50) and 5% noise by (a) OMP (b) BLOOP (c) BP (d) BP-BLOT



Reconstructions of 6 closely spaced targets (min distance = 0.2 Rayleigh length) with unresolved grid (superresolution factor = 100) and 5% noise by (a) OMP (b) BLOOMP (c) BP (d) BP-BLOT

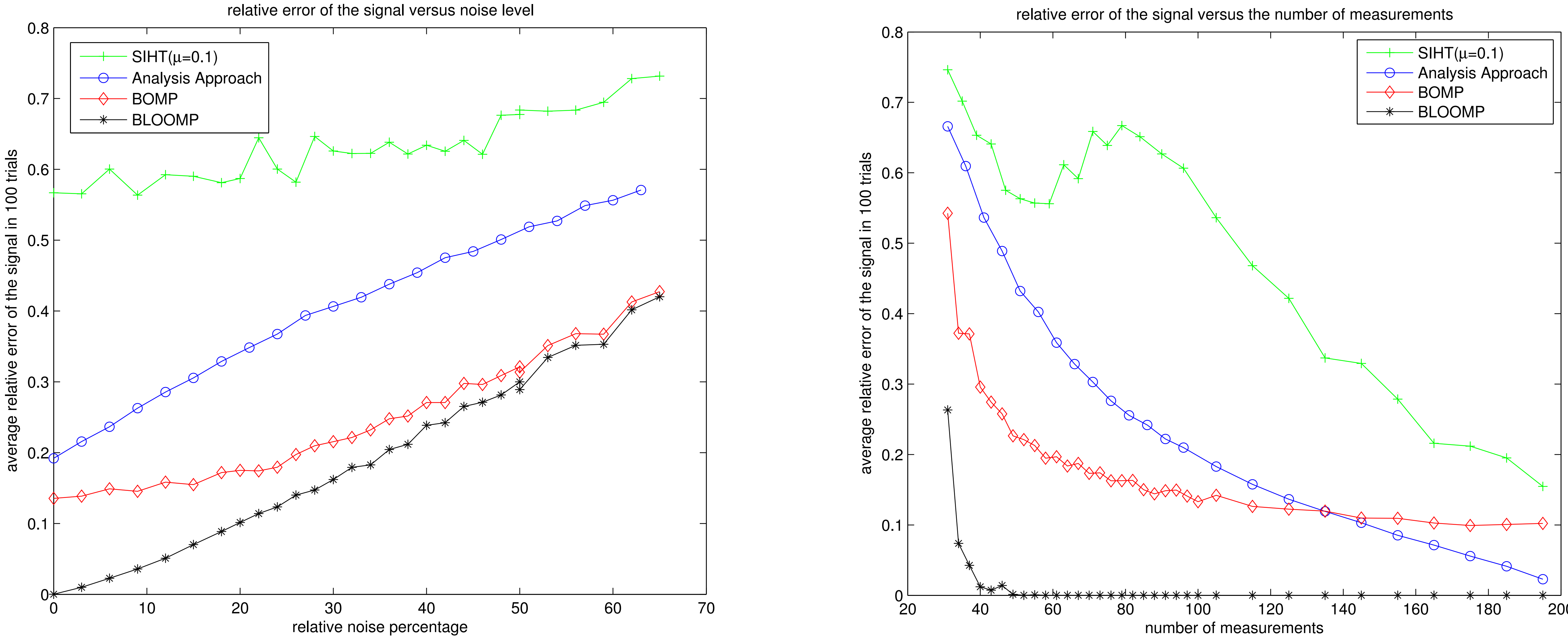
Performance improvement





Success probability versus number of measurements

$$Y = \Phi X + E = ADX + E$$



Relative error versus relative noise (left) and number of data (right)

- Candes, Eldar, Needell, Randall (2011): Frame-adapted BP

$$\min_{\mathbf{z}} \|\mathbf{D}^* \mathbf{z}\|_1, \quad \|\mathbf{A}\mathbf{z} - \mathbf{y}\|_2 \le \varepsilon$$

Assumptions: (i) $\mathbf{A}$ satisfies frame-adapted RIP

(ii) $\mathbf{D}^*\mathbf{z}$ is sparse $\geq$ $\mathbf{s} \times \mathbf{r}$, $r =$ redundancy of dictionary.

- Duarte, Baraniuk (2011): Spectral Compressive Sensing (SIHT)

Conclusion

- Random measurement: sample distribution, illumination
- Effective discretization
- Coherent sensing matrix
- Multiple scattering
- SAR ?