MUSIC for Single-Snapshot Spectral Estimation

Albert Fannjiang, UC Davis

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Collaborator: Wenjing Liao (Duke)

Outline

* Problem formulation
* SS-MUSIC algorithm
* Stability and resolution
* Superresolution
* Conclusion

Spectral Estimation (SE)

* Signal model

$$y^{\varepsilon}(t) = y(t) + \varepsilon(t), \quad y(t) = \sum_{j=1}^{s} x_j e^{-2\pi i \omega_j t}$$

* Problem formulation –

Given *M+1* data $y^{\varepsilon} = [y^{\varepsilon}(t_0), y^{\varepsilon}(t_1), \ldots, y^{\varepsilon}(t_M)]^T$

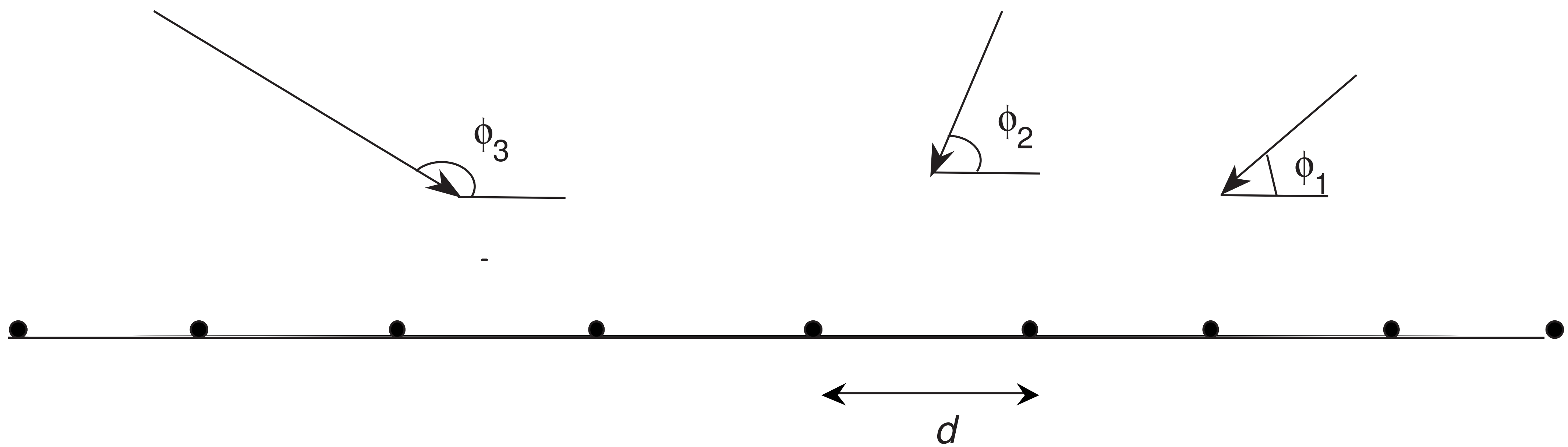
Find $\mathcal{S} = \{\omega_1, \ldots, \omega_s\} \quad x = [x_1, \ldots, x_s]^T$

* Simplest setting – equally spaced samples $t_j = j$

* Frequencies can not be distinguished from their integer-shifted version $\omega \Longleftrightarrow \omega + n, \quad n \in \mathbb{N}$

* Assumption – $\mathcal{S} \subset [0, 1]$ Periodic BC.

Direction Of Arrival (DOA)



* Signal model $$y_k = \sum_{j=1}^{s} x_j e^{-ikd\cos\phi_j/\lambda}, \quad k = 0, 1, \ldots, M$$

* Reduction to spectral estimation – $$\omega_j = d\cos\phi_j/\lambda$$

Fourier analysis

* DFT of order *M+1* – $\mathcal{S} \subset \{\frac{k}{M+1} : k = 0, \ldots, M\}$

  – compressed sensing:

  # data ~ s

* Fourier coefficients of frequency spikes – $k \in \mathbb{Z}$

$$x(\omega) = \sum_{j=1}^{s} x_j \delta(\omega - \omega_j)$$

$$y_k = \int_0^1 x(\omega) e^{-i\omega k} d\omega$$

⇒

$$x(\omega) = \sum_{k\in\mathbb{Z}} y_k e^{i2\pi k\omega}$$

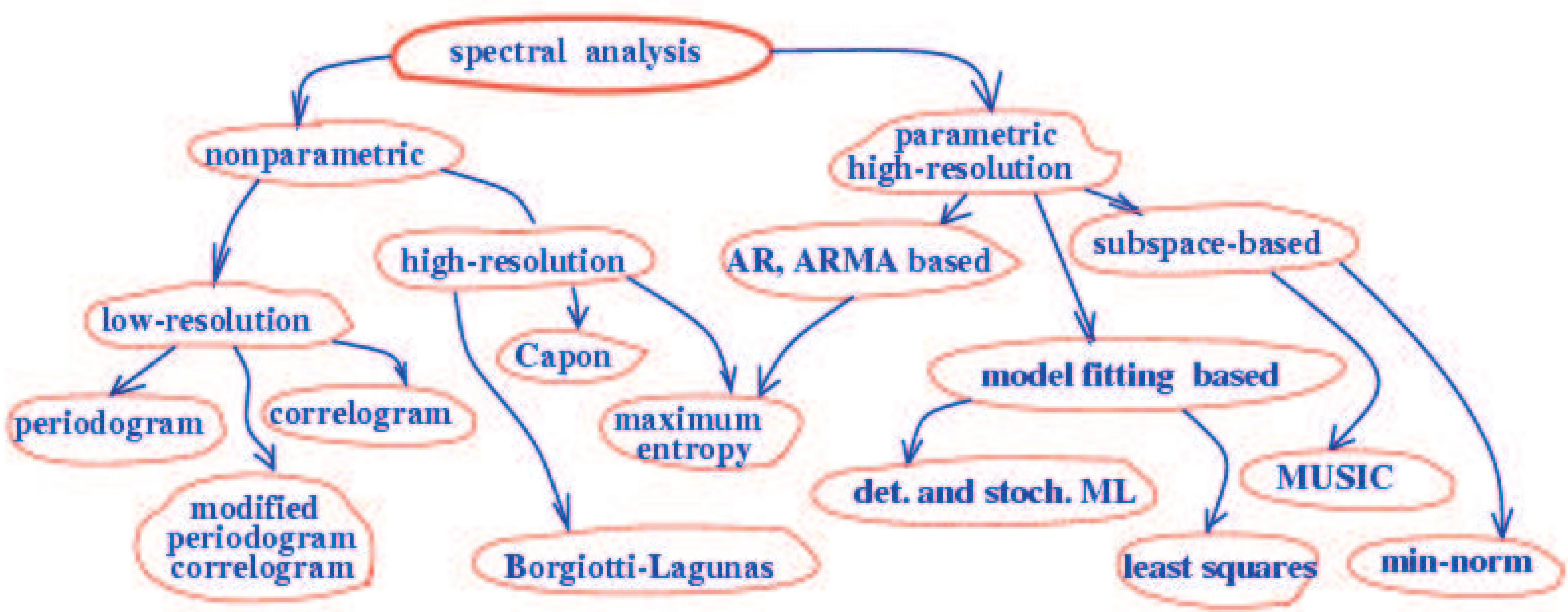
In distributional sense

* SS-SE = FT with *M+1* data points : *M ~ s ?*

* Issues: Resolution, Noise stability

Statistical Signal Processing

Reference: Stoica - Moses 2004



* Single Snapshot measurement ⟶ Deterministic approach

Vector Form

* Vectorization

$$y = [y_k]_{k=0}^{M},\ \varepsilon = [\varepsilon_k]_{k=0}^{M} \text{ and } y^\varepsilon = y + \varepsilon \in \mathbb{C}^{M+1}$$

$$\phi^M(\omega) = [1\ e^{-2\pi i\omega}\ e^{-2\pi i 2\omega}\ \ldots\ e^{-2\pi i M\omega}]^T \in \mathbb{C}^{M+1}$$

$$\Phi^M = [\phi^M(\omega_1)\ \phi^M(\omega_2)\ \ldots\ \phi^M(\omega_s)] \in \mathbb{C}^{(M+1)\times s}$$

* Signal model in vector form $y^\varepsilon = \Phi^M x + \varepsilon.$

Matrix unknown → Nonlinear inverse problem

* Rayleigh resolution length (RL): first zero of the sinc function

$$D(\omega) = \int_{-M/2}^{M/2} e^{2\pi i t\omega} dt = \frac{\sin \pi\omega M}{\pi\omega}$$ → RL = *1/M*

Hankel Data Matrix

$$H = \mathrm{Hankel}(y) = \begin{bmatrix} y_0 & y_1 & \cdots & y_{M-L} \\ y_1 & y_2 & \cdots & y_{M-L+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_L & y_{L+1} & \cdots & y_M \end{bmatrix}$$

* Vandermonde decomposition

$$H = \Phi^L X (\Phi^{M-L})^T, \quad X = \mathrm{diag}(x_1, \ldots, x_s)$$

$$\Phi^L = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ e^{-2\pi i \omega_1} & e^{-2\pi i \omega_2} & \ldots & e^{-2\pi i \omega_s} \\ (e^{-2\pi i \omega_1})^2 & (e^{-2\pi i \omega_2})^2 & \ldots & (e^{-2\pi i \omega_s})^2 \\ \vdots & \vdots & \vdots & \vdots \\ (e^{-2\pi i \omega_1})^L & (e^{-2\pi i \omega_2})^L & \ldots & (e^{-2\pi i \omega_s})^L \end{bmatrix}$$

Signal Model in Matrix Form

$$H^{\varepsilon} = H + E = \Phi^L X (\Phi^{M-L})^T + E$$

* Conditions – $L \geq s, \quad M - L + 1 \geq s \Longleftrightarrow$ *M+1>2s*

* SVD $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \sigma_s > 0$

$$H = [\underbrace{U_1}_{(L+1)\times s} \quad \underbrace{U_2}_{(L+1)\times(L+1-s)}] \underbrace{\mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_s, 0, \ldots, 0)}_{(L+1)\times(M-L+1)} [\underbrace{V_1}_{(M-L+1)\times s} \quad \underbrace{V_2}_{(M-L+1)\times(M-L+1-s)}]^{\star}$$

* Observation – $\mathcal{P}_2 \mathrm{w} = U_2(U_2^{\star}\mathrm{w}), \ \forall \mathrm{w} \in \mathbb{C}^{L+1}$

$$\omega \in \mathcal{S} \text{ if and only if } \mathcal{P}_2 \phi^L(\omega) = \mathbf{0}$$

* Noise-space correlation

$$R(\omega) = \frac{\|\mathcal{P}_2 \phi^L(\omega)\|_2}{\|\phi^L(\omega)\|_2} = \frac{\|U_2^{\star} \phi^L(\omega)\|_2}{\|\phi^L(\omega)\|_2}$$

MUSIC with Noisy Data

$$H^\varepsilon = [\underbrace{U_1^\varepsilon}_{(L+1)\times s} \quad \underbrace{U_2^\varepsilon}_{(L+1)\times(L+1-s)}] \underbrace{\mathrm{diag}(\sigma_1^\varepsilon, \sigma_2^\varepsilon, \ldots, \sigma_s^\varepsilon, \sigma_{s+1}^\varepsilon, \ldots)}_{(L+1)\times(M-L+1)} [\underbrace{V_1^\varepsilon}_{(M-L+1)\times s} \quad \underbrace{V_2^\varepsilon}_{(M-L+1)\times(M-L+1-s)}]^\star$$

(Weyl's Theorem). $|\sigma_j^\varepsilon - \sigma_j| \le \|E\|_2, \; j = 1, 2, \ldots$

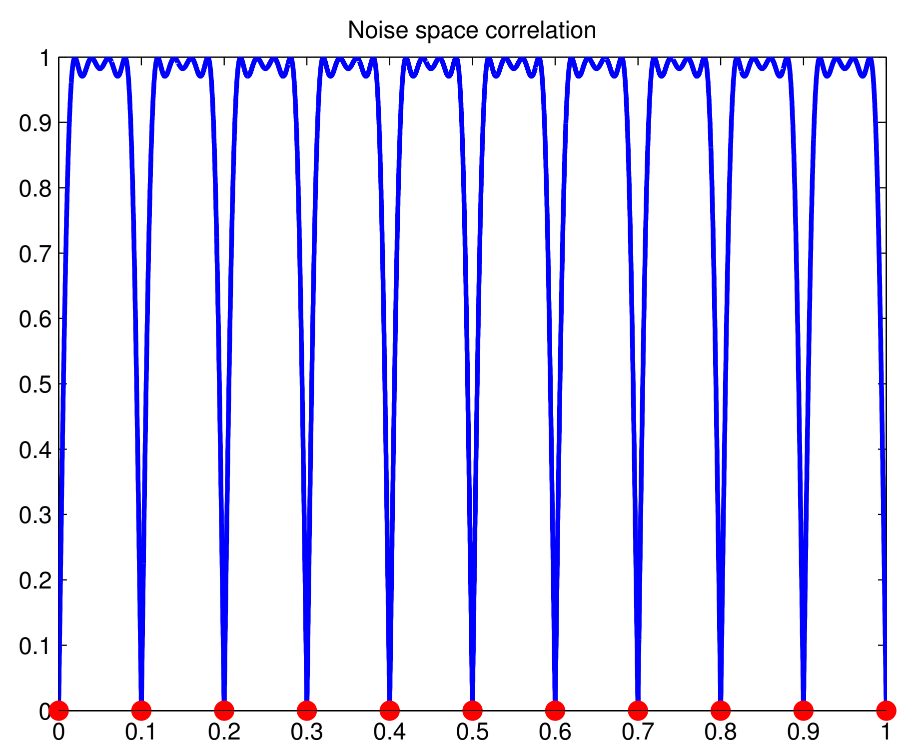
Input: $y^\varepsilon \in \mathbb{C}^{M+1}, s, L$.

1) Form matrix $H^\varepsilon = \mathrm{Hankel}(y^\varepsilon) \in \mathbb{C}^{(L+1)\times(M-L+1)}$.

2) SVD: $H^\varepsilon = [U_1^\varepsilon \; U_2^\varepsilon]\mathrm{diag}(\sigma_1^\varepsilon, \ldots, \sigma_s^\varepsilon, \ldots)[V_1^\varepsilon \; V_2^\varepsilon]^\star$, where $U_1^\varepsilon \in \mathbb{C}^{(L+1)\times s}$.

3) Compute correlation function $R^\varepsilon(\omega) = \|U_2^{\varepsilon\star}\phi^L(\omega)\|_2 / \|\phi^L(\omega)\|_2$.

Output: $\hat{\mathcal{S}} = \{\omega$ corresponding to s largest local maxima of $J^\varepsilon(\omega)\}$.



Stability Analysis

$$|R^{\varepsilon}(\omega) - R(\omega)| \leq \alpha \|E\|_2, \quad \alpha = \frac{4\sigma_1 + 2\|E\|_2}{(\sigma_s - \|E\|_2)^2}$$

Noise tolerance ~ min SV of H

* I.I.D. noise

spectral norm of E ~ $\sqrt{M \log M}$ for $M \gg 1$

* Claim – max and min SV of H ~ $M/2$

$$H = \Phi^L X (\Phi^{M-L})^T, \quad X = \mathrm{diag}(x_1, \ldots, x_s)$$

Discrete Ingham Inequalities

If

$$q = \min_{j \neq l} d(\omega_j, \omega_l) > \frac{1}{L}\sqrt{\frac{2}{\pi}}\left(\frac{2}{\pi} - \frac{4}{L}\right)^{-\frac{1}{2}}$$

then

$$\left(\frac{2}{\pi} - \frac{2}{\pi L^2 q^2} - \frac{4}{L}\right)\|\mathrm{c}\|_2^2 \leq \frac{1}{L}\|\Phi^L \mathrm{c}\|_2^2 \leq \left(\frac{4\sqrt{2}}{\pi} + \frac{\sqrt{2}}{\pi L^2 q^2} + \frac{3\sqrt{2}}{L}\right)\|\mathrm{c}\|_2^2, \ \forall \mathrm{c} \in \mathbb{C}^s$$

$$\frac{1}{L}\sigma_{\max}^2(\Phi^L) \leq \frac{4\sqrt{2}}{\pi} + \frac{\sqrt{2}}{\pi L^2 q^2} + \frac{3\sqrt{2}}{L}$$

$$\frac{1}{L}\sigma_{\min}^2(\Phi^L) \geq \frac{2}{\pi} - \frac{2}{\pi L^2 q^2} - \frac{4}{L}.$$

Discrete Ingham Inequalities

If $q = \min_{j \neq l} d(\omega_j, \omega_l) > \max\left(\frac{1}{L}\sqrt{\frac{2}{\pi}}\left(\frac{2}{\pi} - \frac{4}{L}\right)^{-\frac{1}{2}}, \frac{1}{M-L}\sqrt{\frac{2}{\pi}}\left(\frac{2}{\pi} - \frac{4}{M-L}\right)^{-\frac{1}{2}} \right)$

then

$$\frac{\sigma_s^2}{L(M-L)} \geq x_{\min}^2 \left(\frac{2}{\pi} - \frac{2}{\pi L^2 q^2} - \frac{4}{L}\right)\left(\frac{2}{\pi} - \frac{2}{\pi (M-L)^2 q^2} - \frac{4}{M-L}\right)$$

$$\frac{\sigma_1^2}{L(M-L)} \leq x_{\max}^2 \left(\frac{4\sqrt{2}}{\pi} + \frac{\sqrt{2}}{\pi L^2 q^2} + \frac{3\sqrt{2}}{L}\right)\left(\frac{4\sqrt{2}}{\pi} + \frac{\sqrt{2}}{\pi (M-L)^2 q^2} + \frac{3\sqrt{2}}{M-L}\right)$$

* $L = M/2$ & the spacing > 2 RL

Max and Min SV are on the same order $M/2$

Figures of Merit

Hausdorff Distance (HD)

$$d(\hat{\mathcal{S}}, \mathcal{S}) = \max \left\{ \max_{\hat{\omega} \in \hat{\mathcal{S}}} \min_{\omega \in \mathcal{S}} d(\hat{\omega}, \omega) \ , \ \max_{\omega \in \mathcal{S}} \min_{\hat{\omega} \in \hat{\mathcal{S}}} d(\hat{\omega}, \omega) \right\}$$

$$\text{Noise-to-Signal Ratio (NSR)} = \mathbb{E}(\|\varepsilon\|_2)/\|y\|_2 = \sigma\sqrt{2(M+1)}/\|y\|_2$$

CPU Run Time

Dynamic Range

BLOOMP

* F-Liao 2012

Algorithm 3. Band-excluded Locally Optimized Orthogonal Matching Pursuit (BLOOMP).

Input: $\mathbf{A}, \mathbf{b}, \eta > 0$
Initialization: $\mathbf{x}^0 = 0, \mathbf{r}^0 = \mathbf{b}$, and $S^0 = \emptyset$
Iteration: For $n = 1, \ldots, s$

1. $i_{\max} = \arg\max_i |\langle \mathbf{r}^{n-1}, \mathbf{a}_i \rangle|, i \notin B_\eta^{(2)}(S^{n-1})$.
2. $S^n = \mathrm{LO}(S^{n-1} \cup \{i_{\max}\})$, where LO is the output of Algorithm 2.
3. $\mathbf{x}^n = \arg\min_{\mathbf{z}} \|\mathbf{Az} - \mathbf{b}\|_2$ s.t. $\mathrm{supp}(\mathbf{z}) \in S^n$.
4. $\mathbf{r}^n = \mathbf{b} - \mathbf{A}\mathbf{x}^n$.

Output: $\mathbf{x}^s$.

Algorithm 2. Local Optimization (LO).

Input: $\mathbf{A}, \mathbf{b}, \eta > 0,\ S^0 = \{i_1, \ldots, i_k\}$
Iteration: For $n = 1, 2, \ldots, k$

1. $\mathbf{x}^n = \arg\min_{\mathbf{z}} \|\mathbf{Az} - \mathbf{b}\|_2, \quad \mathrm{supp}(\mathbf{z}) = (S^{n-1} \backslash \{i_n\}) \cup \{j_n\},\ j_n \in B_\eta(\{i_n\})$.
2. $S^n = \mathrm{supp}(\mathbf{x}^n)$.

Output: S^k.

BP/SDP

*Tang et al. 2013, Candes-Fernandez-Granda 2013

$$\min \|x\|_{\mathrm{TV}}, \quad \|y^{\varepsilon} - y\|_2 \le \epsilon$$

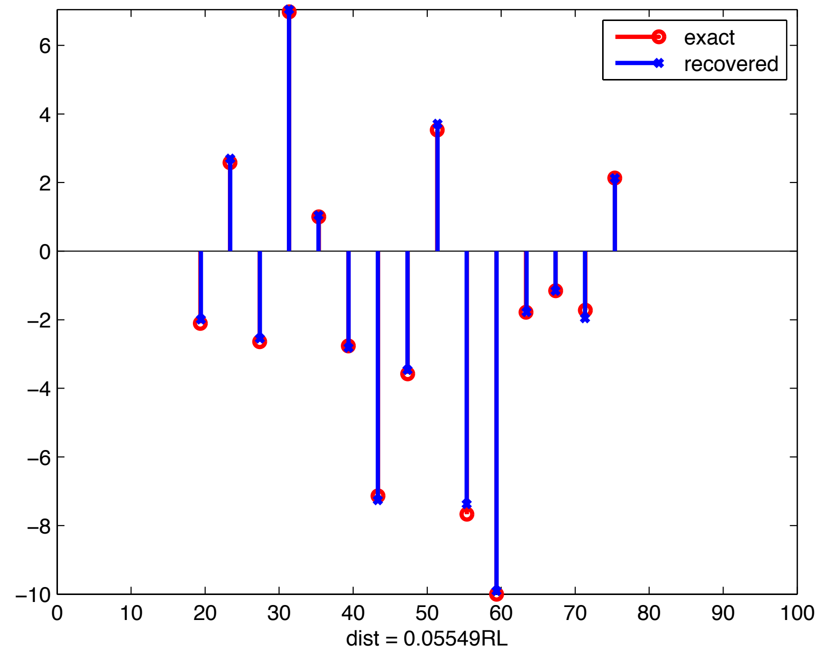
$$y = \text{inverse Fourier transform of } x$$

Stability requirement: 4 RL

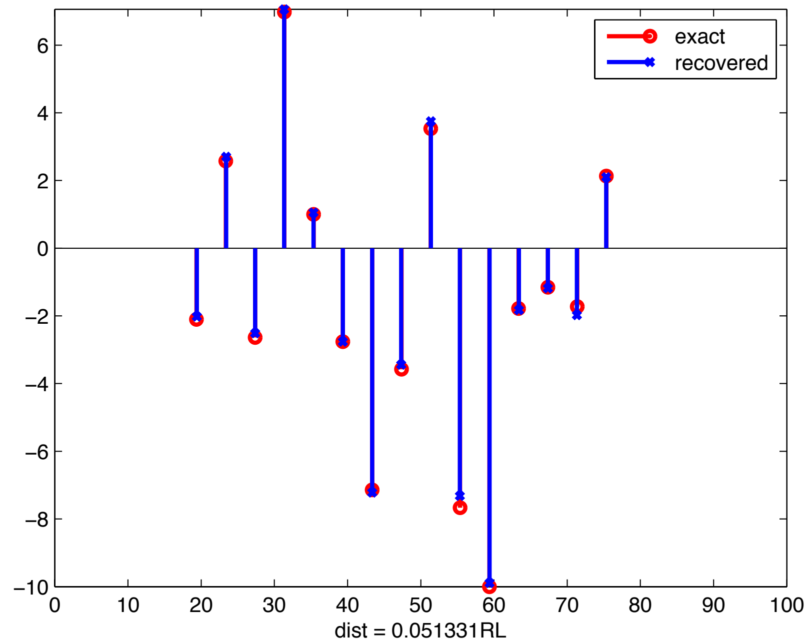
Code: http://www.stanford.edu/~cfgranda/superres_sdp_noisy.m SDP is solved in CVX. Output of SDP is the dual solution of TV minimization. Frequencies are identified through root findings of a polynomial and amplitudes are solved through least squares.

Band Excluded Thresholding (BET)
Input: $\tilde{\omega}, \tilde{x}, s, r$ (radius of excluded band).
Initialization: $\hat{\omega} = [\,]$.
Iteration: for $k = 1, \ldots, s$
1) Find j such that $\lvert\tilde{x}_j\rvert = \max_i \lvert\tilde{x}_i\rvert$.
If $\tilde{x}_j = 0$, then go to **Output**.
2) Update the support vector: $\hat{\omega} = [\hat{\omega}\ ;\ \tilde{\omega}_j]$.
3) For $i = 1 : n$
If $\tilde{\omega}_i \in (\tilde{\omega}_j - r, \tilde{\omega}_j + r)$, set $\tilde{x}_i = 0$.
Output: $\hat{\omega}$.

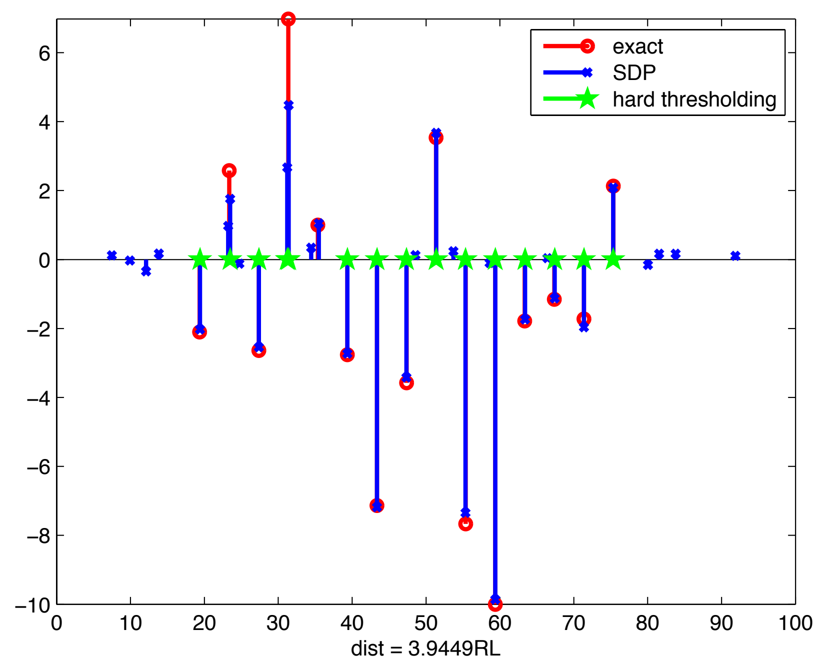
Reconstruction of 15 real-valued amplitudes separated by 4RL. Dynamic range = 10 and NSR = 10%.



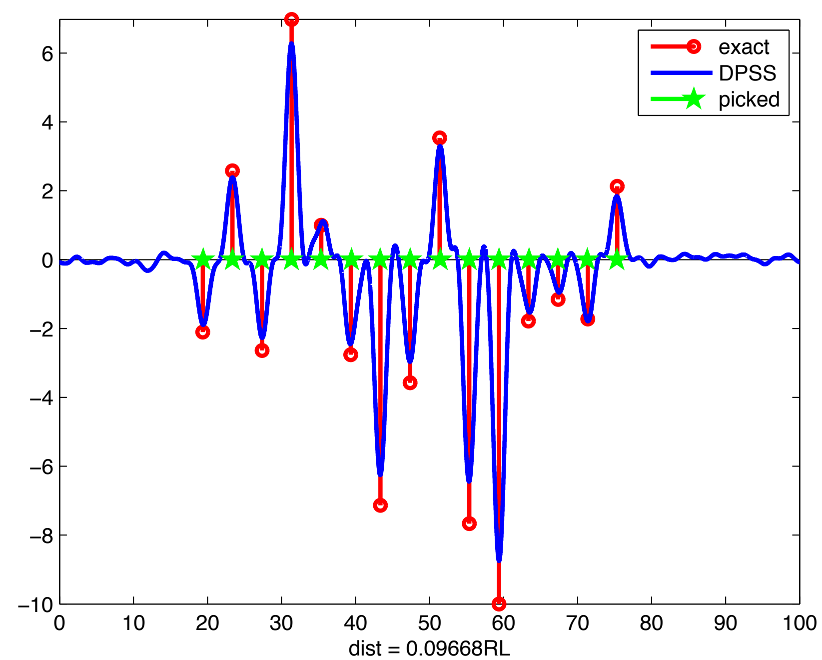
(a) MUSIC. Red: exact; Blue: recovered. $d(\hat{\mathcal{S}}, \mathcal{S}) \approx 0.06$RL.



(b) BLOOMP. Red: exact; Blue: recovered. $d(\hat{\mathcal{S}}, \mathcal{S}) \approx 0.05$RL.



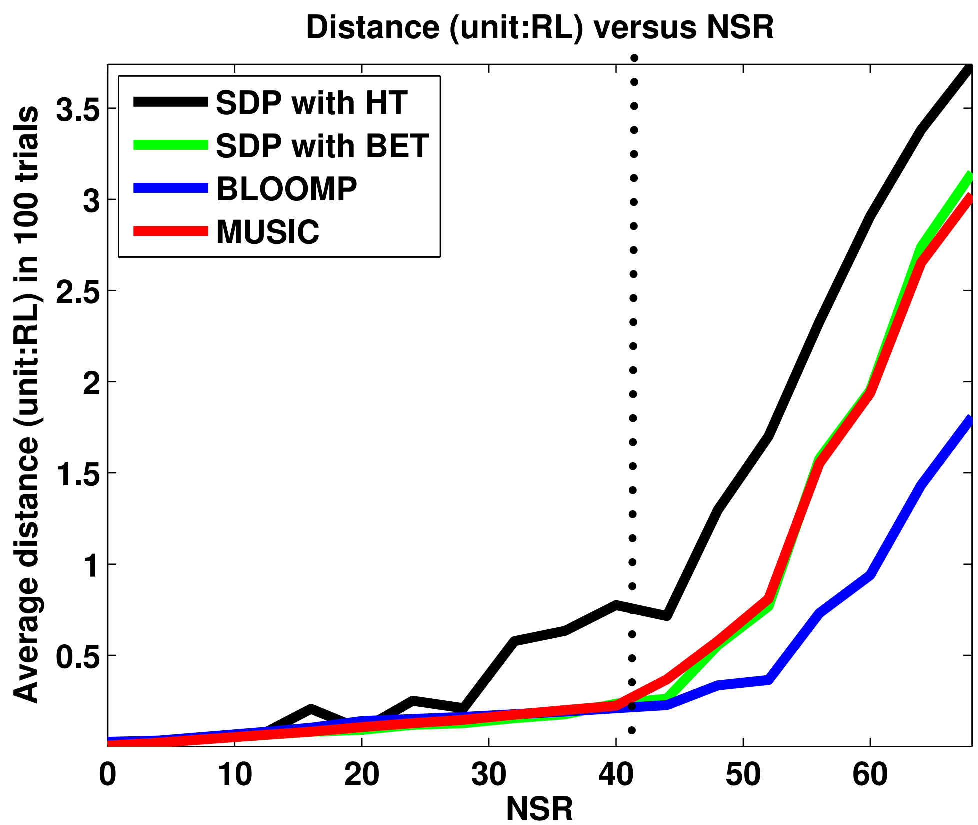
(c) SDP. Red: exact; Blue: Primal solution of SDP. Hard thresholding (green) yields $d(\hat{\mathcal{S}}, \mathcal{S}) \approx$ 3.94RL. The true amplitude around 33RL is recovered as two amplitudes and the BET technique can be used to eliminate the smaller one in the step of frequency selection.



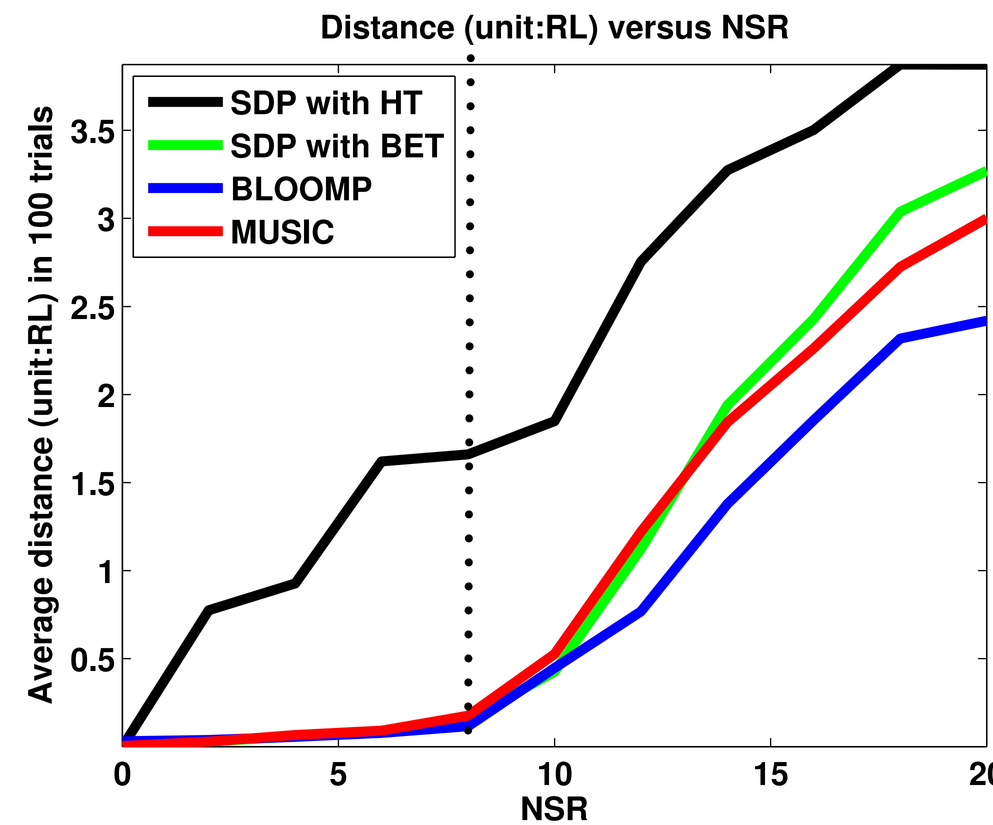
(d) MF using prolates. Red: exact; Blue: inverse Fourier transform of y^{ε} windowed by the first DPSS sequence; Green: frequencies selected by the BLO technique. $d(\hat{\mathcal{S}}, \mathcal{S}) \approx 0.10$RL.

HD versus NSR

Frequencies separated between 4RL and 5RL



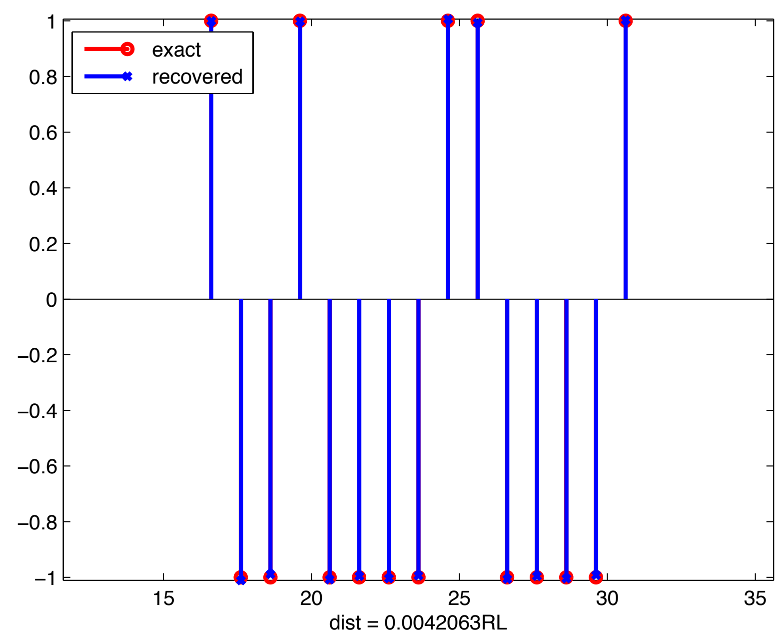
(a) Dynamic range = 1. Average running time for SDP and MUSIC in one experiment is 20.3583s and 0.3627s while the average running time for BLOOMP is 6.3420s ($F = 20$), 3.2788s ($F = 10$) and 1.7610($F = 5$).



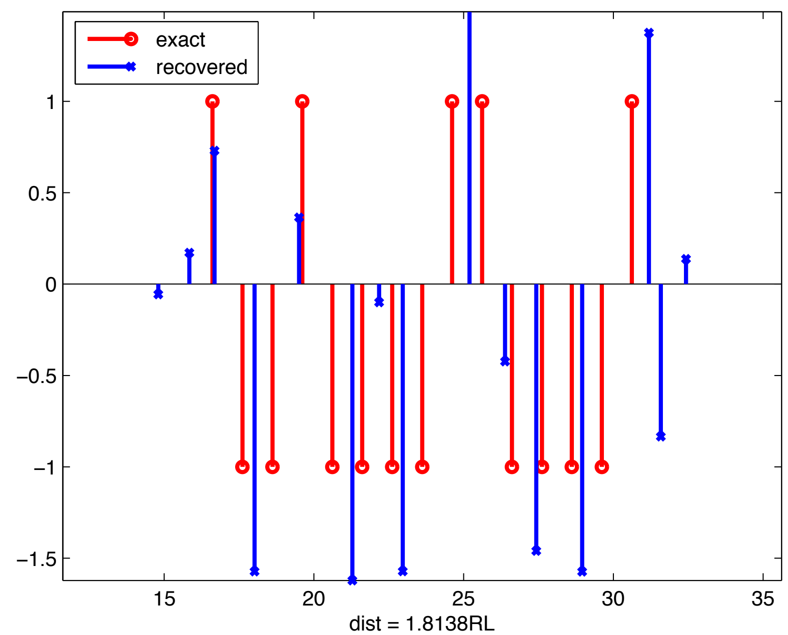
(b) Dynamic range = 10. Average running time for SDP and MUSIC in one experiment is 20.5913s and 0.3661s while the average running time for BLOOMP is 6.2623s ($F = 20$), 3.3030s ($F = 10$) and 1.7542s ($F = 5$).

Linear stability: (i) 0 ~ 0.2 HD; (ii) >0.2 HD

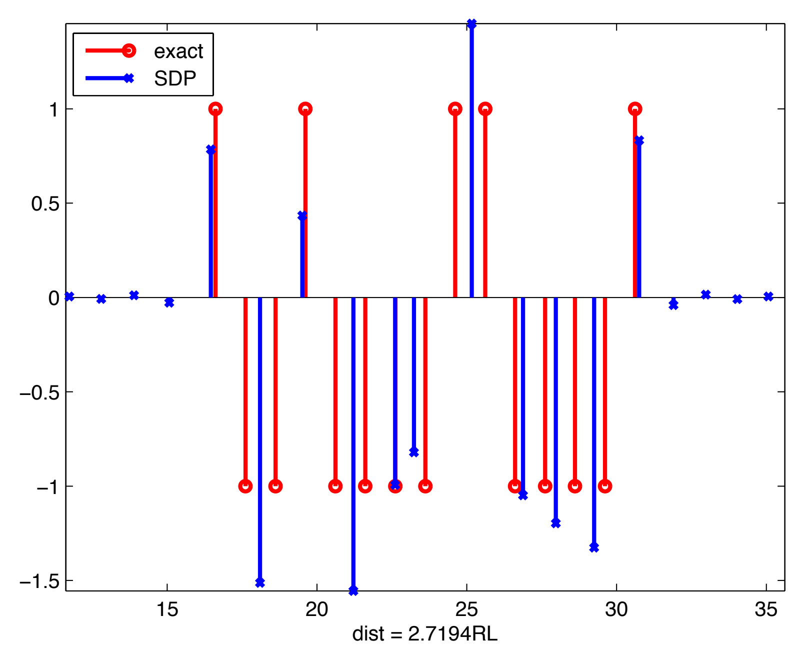
Reconstruction of 15 real-valued frequencies separated by 1RL. Dynamic range = 1 and NSR = 0%.



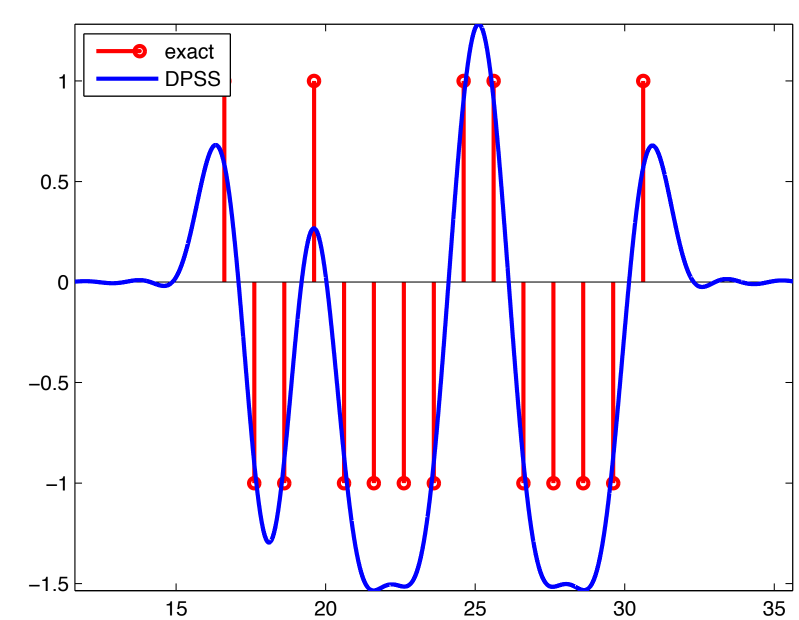
(a) MUSIC. Red: exact; Blue: recovered. $d(\hat{\mathcal{S}}, \mathcal{S}) \approx 0.004$RL.



(b) BLOOMP. Red: exact; Blue: recovered. $d(\hat{\mathcal{S}}, \mathcal{S}) \approx 1.81$RL.



(c) SDP. Red: exact; Blue: recovered. Hard thresholding yields $d(\hat{\mathcal{S}}, \mathcal{S}) \approx 2.72$RL.



(d) MF using prolates. Red: exact; Blue: inverse Fourier transform of y windowed by the first DPSS sequence.

Superresolution

* MUSIC ~ 2 RL → Linear stability

* Superresolution (Donoho 92) – with unresolved grid

Assumption $\mathcal{S} \subset \mathcal{L}(\Delta)$ $\mathcal{L}(\Delta) = \{k\Delta, k \in \mathbb{Z}\}$

Suppose R_* *is the least positive integer such that*

$$|\omega_{j+R} - \omega_j| > R_*/M, \ \forall j.$$

Continuous data $t \in [0, M]$

min SV $$\nu(\Delta, M, R)\|\mathrm{c}\|_2^2 \le \int_0^M |\sum_{\omega_j \in \mathcal{S}} \mathrm{c}_j e^{-2\pi i \omega_j t}|^2 dt$$

Lower bound $$\nu(\Delta, M, R) \ge \Delta^{2R+1}\alpha(M, R)$$

Upper bound $$\nu(\Delta, M, R) \le \Delta^{2R-1}\beta(M, R)$$

Optimal Superresolution

Lattice spacing Δ ~ minimum separation of frequencies

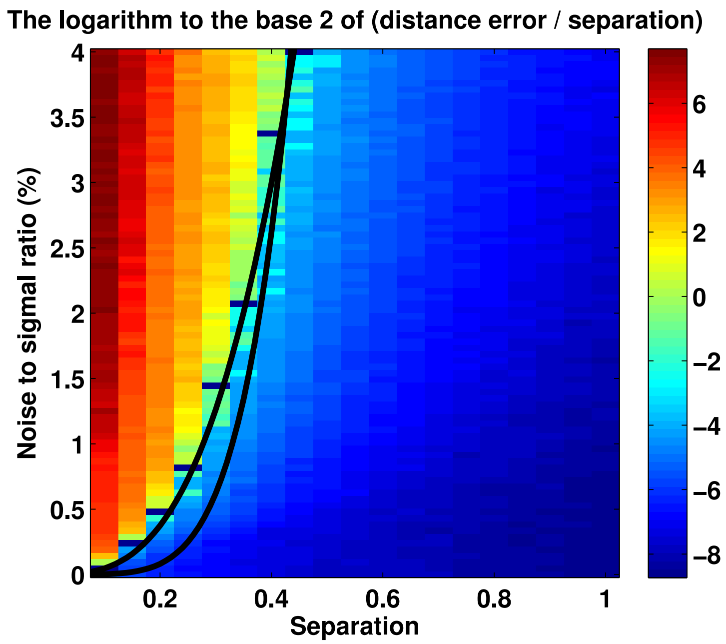
Rayleigh Index (RI) ~ R = Size of largest cluster

$$|\omega_{j+R} - \omega_j| > R_*/M, \ \forall j.$$

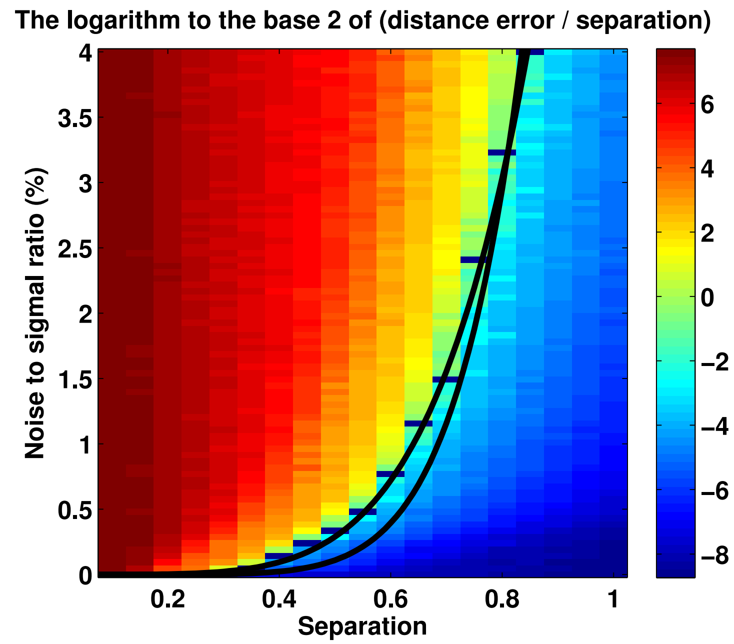
$$|R^\varepsilon(\omega) - R(\omega)| \le \alpha \|E\|_2, \qquad \alpha = \frac{4\sigma_1 + 2\|E\|_2}{(\sigma_s - \|E\|_2)^2}$$

⇒ Noise tolerance ~ min SV $\nu(\Delta, M, R) \le \Delta^{2R-1}\beta(M, R)$

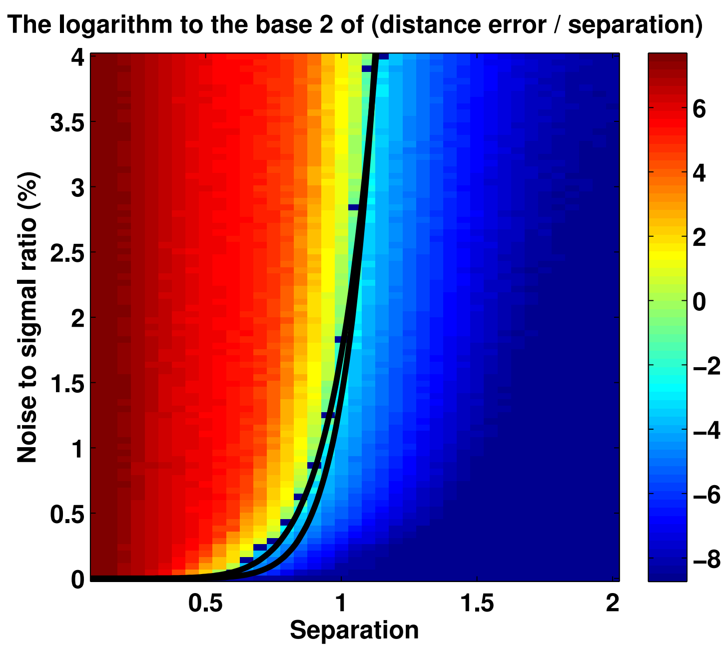
Numerical performance achieves the upper bound power law !



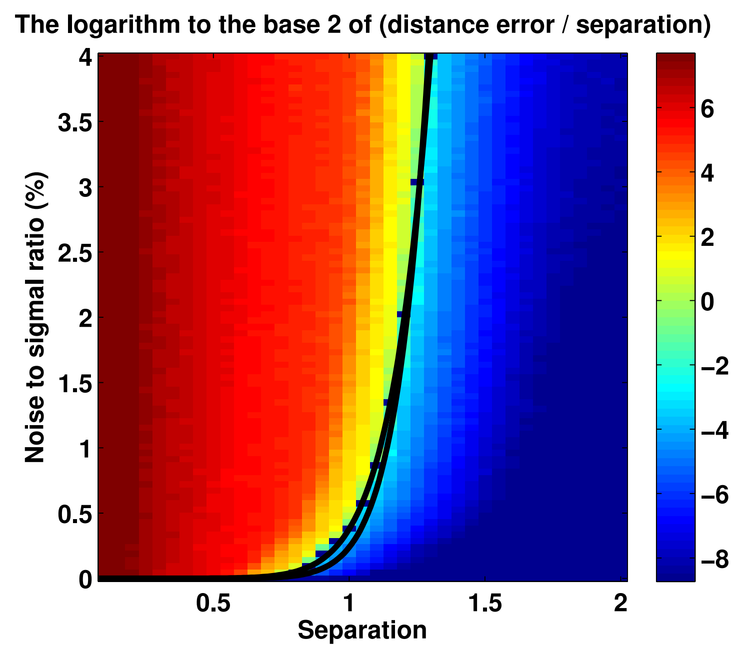
(a) Two-point resolution of MUSIC, $R = 2$



(b) Three-point resolution of MUSIC, $R = 3$



(c) Four-point resolution of MUSIC, $R = 4$



(d) Five-point resolution of MUSIC, $R = 5$

In each cases we plot two black curves $y = cx^k$ corresponding to $k = 2R - 1$ and $k = 2R + 1$ and with a proper choice of c such that the curve fits all transition points under least squares. In (a) the top is $y = 46.7828x^3$ and the bottom curve is $y = 255.677x^5$. In (b) the top is $y = 9.2532x^5$ and the bottom curve is $y = 14.0662x^7$. In (c) the top is $y = 1.7191x^7$ and the bottom curve is $y = 1.3906x^9$. In (d) the top is $y = 0.3790x^9$ and the bottom curve is $y = 0.2384x^11$. Interestingly the phase transition curve fits very well with the top curve $y = cx^{2R-1}$ in all cases.

Conclusion

* Deterministic single-snapshot MUSIC
* Discrete Ingham Inequalities
* Stability and Resolution –
  linear stability for separation > 2 RL
* Efficiency
* Nearly optimal superresolution