Phase Retrieval with Random Mask

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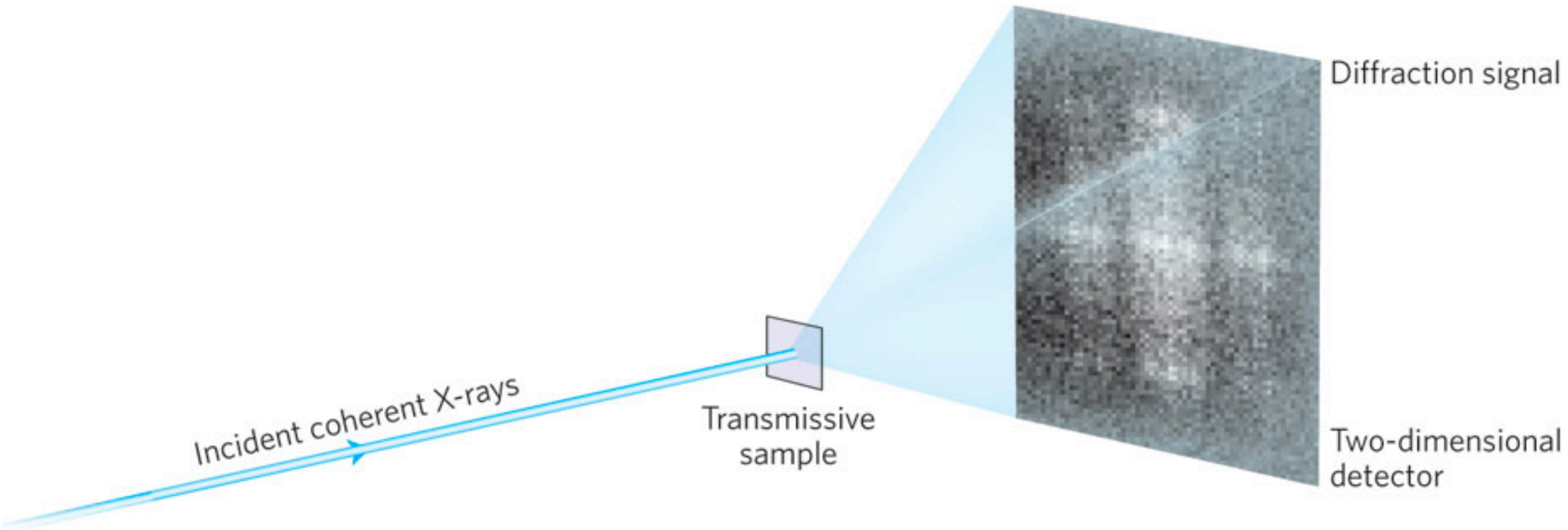
Lensless coherent imaging



Phase retrieval with oversampling

Discrete finite objects

Let $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$ and $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$.

$$\text{multi-index}: \quad \mathbf{z}^{\mathbf{n}} = z_1^{n_1} z_2^{n_2}$$

Let the object be represented by $f(\mathbf{n}), \mathbf{n} \le \mathbf{N} = (N, N)$

Fourier transform describes wave propagation

$$F(e^{i2\pi w_1}, e^{i2\pi w_2}) = \sum_{\mathbf{n}} f(\mathbf{n}) e^{-i2\pi \mathbf{n}\cdot\mathbf{w}}$$

Analytic continuation $\Longrightarrow$ z-transform

$$F(\mathbf{z}) = \sum_{\mathbf{n}} f(\mathbf{n}) \mathbf{z}^{-\mathbf{n}}.$$

Discrete phase retrieval problem:

Determine $f(\mathbf{n})$ from Fourier magnitude data

$$|F(\mathbf{w})|, \quad \forall \mathbf{w} = (e^{i2\pi w_1}, e^{i2\pi w_2}) \in [0, 1]^2$$

Phase = Face ?

$$f_L = \text{Lena}$$
$$F_L(\mathbf{w}) = |F_L(\mathbf{w})| e^{\mathrm{i}\theta_L(\mathbf{w})}$$

$$F_1(\mathbf{w}) = |F_B(\mathbf{w})| e^{\mathrm{i}\theta_L(\mathbf{w})}$$
$$f_1 = |\mathbf{\Phi}^* F_1|$$

$$f_B = \text{Barbara}$$
$$F_B(\mathbf{w}) = |F_B(\mathbf{w})| e^{\mathrm{i}\theta_B(\mathbf{w})}$$

$$F_2(\mathbf{w}) = |F_L(\mathbf{w})| e^{\mathrm{i}\theta_B(\mathbf{w})}$$
$$f_2 = |\mathbf{\Phi}^* F_2|$$

Phase = Face !

$$F_1(\mathbf{w}) = |F_B(\mathbf{w})| e^{\mathrm{i}\theta_L(\mathbf{w})}$$
$$f_1 = |\mathbf{\Phi}^* F_1|$$

$$F_2(\mathbf{w}) = |F_L(\mathbf{w})| e^{\mathrm{i}\theta_B(\mathbf{w})}$$
$$f_2 = |\mathbf{\Phi}^* F_2|$$

Fourier magnitude data:

$$|F(\mathbf{w})|^2 = \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \sum_{\mathbf{m}} f(\mathbf{m}+\mathbf{n}) f^*(\mathbf{m}) e^{-\mathrm{i}2\pi\mathbf{n}\cdot\mathbf{w}}$$
$$= \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \mathcal{C}_f(\mathbf{n}) e^{-\mathrm{i}2\pi\mathbf{n}\cdot\mathbf{w}}$$

where

$$\mathcal{C}_f(\mathbf{n}) = \sum_{\mathbf{m}} f(\mathbf{m}+\mathbf{n}) f^*(\mathbf{m})$$

is the autocorrelation function of f.

Fourier magnitude data contain complete information about autocorrelation function.

Sampling Theorem:
supp$(C_f) \subset [-N, N]^2 \Longrightarrow [0,1]^2$ **is reduced to the Nyquist grid**

$$\mathcal{M} = \left\{ (k_1, k_2) : k_j = 0, \frac{1}{2N+1}, \frac{2}{2N+1}, \cdots, \frac{2N}{2N+1} \right\}$$

Harmonic (50%) & non-harmonic (50%) Fourier coefficients

Oversampling ratio

$$\sigma = \frac{\text{Fourier magnitude data number}}{\text{unknown image pixel number}}$$

Standard ratio: $\sigma = 2^d$

Compressed measurement: $\sigma < 2^d$

Trivial ambiguities

Autocorrelation:

$$\mathcal{C}_f(\mathbf{n}) = \sum_{\mathbf{m}+\mathbf{n}\in\mathcal{N}} f(\mathbf{m}+\mathbf{n}) f^*(\mathbf{m})$$

Invariant under:

(i) global phase,

$$f(\mathbf{n}) \longrightarrow e^{\mathrm{i}\theta} f(\mathbf{n}), \quad \textbf{for some } \theta \in [0, 2\pi],$$

(ii) spatial translation

$$f(\mathbf{n}) \longrightarrow f(\mathbf{n}+\mathbf{m}), \quad \textbf{some } \mathbf{m} \in \mathbb{Z}^2$$

(iii) conjugate inversion (twin image)

$$f(\mathbf{n}) \longrightarrow f^*(\mathbf{N}-\mathbf{n}).$$

Nontrivial ambiguity

THEOREM (Hayes 82, Pitts-Greenleaf 03)

Let the z-transform $F(\mathbf{z})$ of a finite complex-valued sequence $\{f(\mathbf{n})\}$ be given by

$$F(\mathbf{z}) = \alpha \mathbf{z}^{-\mathbf{m}} \prod_{k=1}^{p} F_k(\mathbf{z}), \qquad \mathbf{m} \in \mathbb{N}^2, \alpha \in \mathbb{C}$$

where $F_k, k = 1, ..., p$ are nontrivial irreducible polynomials. Let $G(\mathbf{z})$ be the $\mathbf{z}$-transform of another finite sequence $g(\mathbf{n})$. Suppose $|F(\mathbf{w})| = |G(\mathbf{w})|, \forall \mathbf{w} \in [0,1]^2$. Then $G(\mathbf{z})$ must have the form

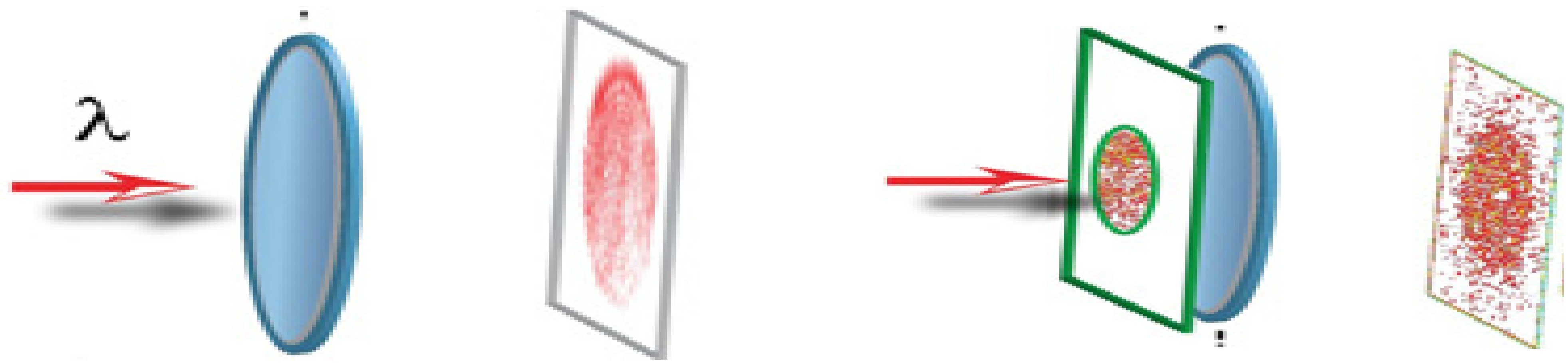
$$G(\mathbf{z}) = |\alpha| e^{\mathrm{i}\theta} \mathbf{z}^{-\mathbf{p}} \left(\prod_{k \in I} F_k(\mathbf{z}) \right) \left(\prod_{k \in I^c} F_k^*(1/\mathbf{z}^*) \right), \qquad \mathbf{p} \in \mathbb{N}^2, \theta \in \mathbb{R}$$

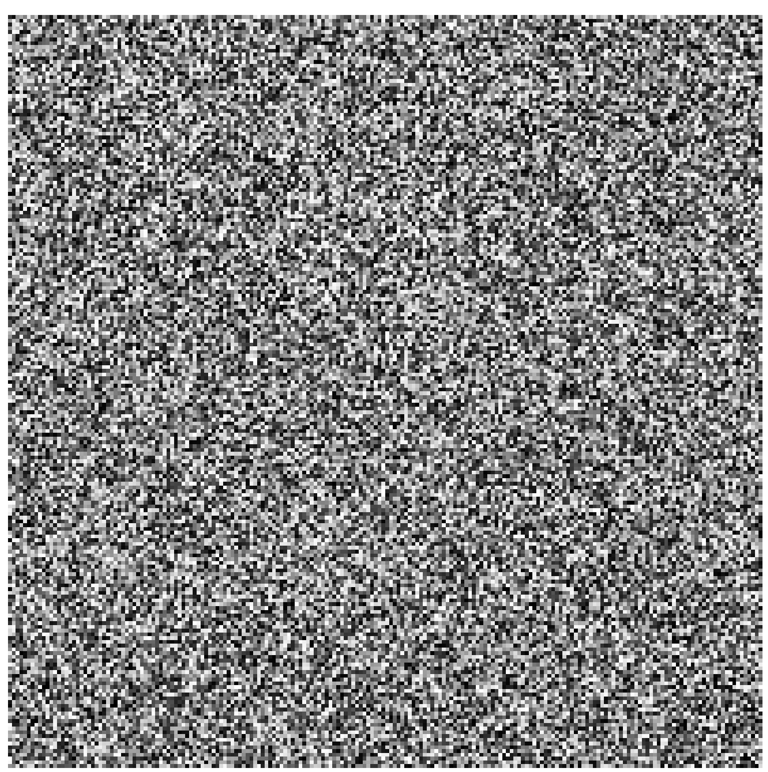
where I is a subset of $\{1, 2, ..., p\}$.

Nontrivial ambiguity: Partial conjugate inversion on factors.

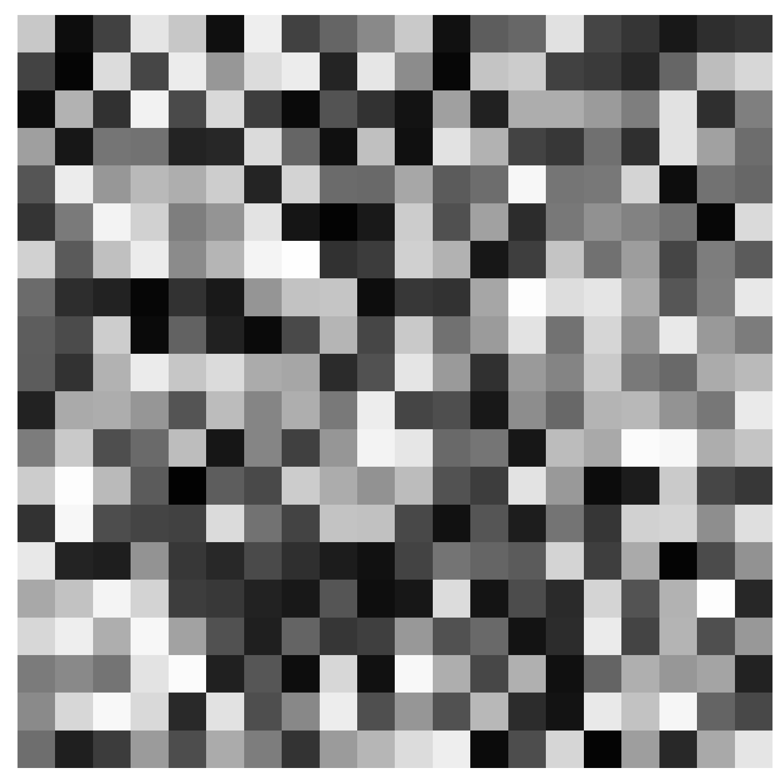
Random mask

Coded aperture imaging





(a) 200×200 HRM



(b) 20×20 LRM

Random illumination

$$\tilde{f}(\mathbf{n}) = f(\mathbf{n})\lambda(\mathbf{n}) \quad \text{(illuminated object)}$$

$\lambda(\mathbf{n})$, representing the illumination field, is a known sequence of samples of random variables.

Let $\lambda(\mathbf{n})$ be continuous random variables with respect to the Lebesque measure on $\mathbb{S}^1$ (the unit circle), $\mathbb{R}$ or $\mathbb{C}$.

Case of $\mathbb{S}^1$ can be facilitated by a random phase modulator with

$$\lambda(\mathbf{n}) = e^{\mathrm{i}\phi(\mathbf{n})}$$

where $\phi(\mathbf{n})$ are continuous random variables on $[0, 2\pi]$.
Case of $\mathbb{R}$: random amplitude modulator.
Case of $\mathbb{C}$: both phase and amplitude modulations.

Irreducibility

THEOREM. Suppose the object $\{f(\mathrm{n})\}$ is rank ≥ 2. Then the the z-transform of the illuminated object $f(\mathrm{n})\lambda(\mathrm{n})$ is irreducible with probability one.

False for rank 1 objects: Fundamental Thm of Algebra

$$\sigma = \frac{\text{Fourier magnitude data number}}{\text{unknown image pixel number}} = 2^d$$

Absolute uniqueness

THEOREM If $f(\mathrm{n})$ is real and nonnegative for every n then, with probability one, f is determined absolutely uniquely by the Fourier magnitude measurement on the lattice $\mathcal{L}$.

THEOREM Suppose the phases of the object belong to $[a, b] \subset [0, 2\pi]$. Then the solution to the Fourier phasing problem has a unique solution with probability

$$1 - |\mathcal{N}| \left| \frac{b-a}{2\pi} \right|^{[S/2]}.$$

Objects w/o constraint

<u>THEOREM</u> Suppose that $\{\lambda_1(\mathrm{n})\}$ are i.i.d. In addition, if either of the following holds true

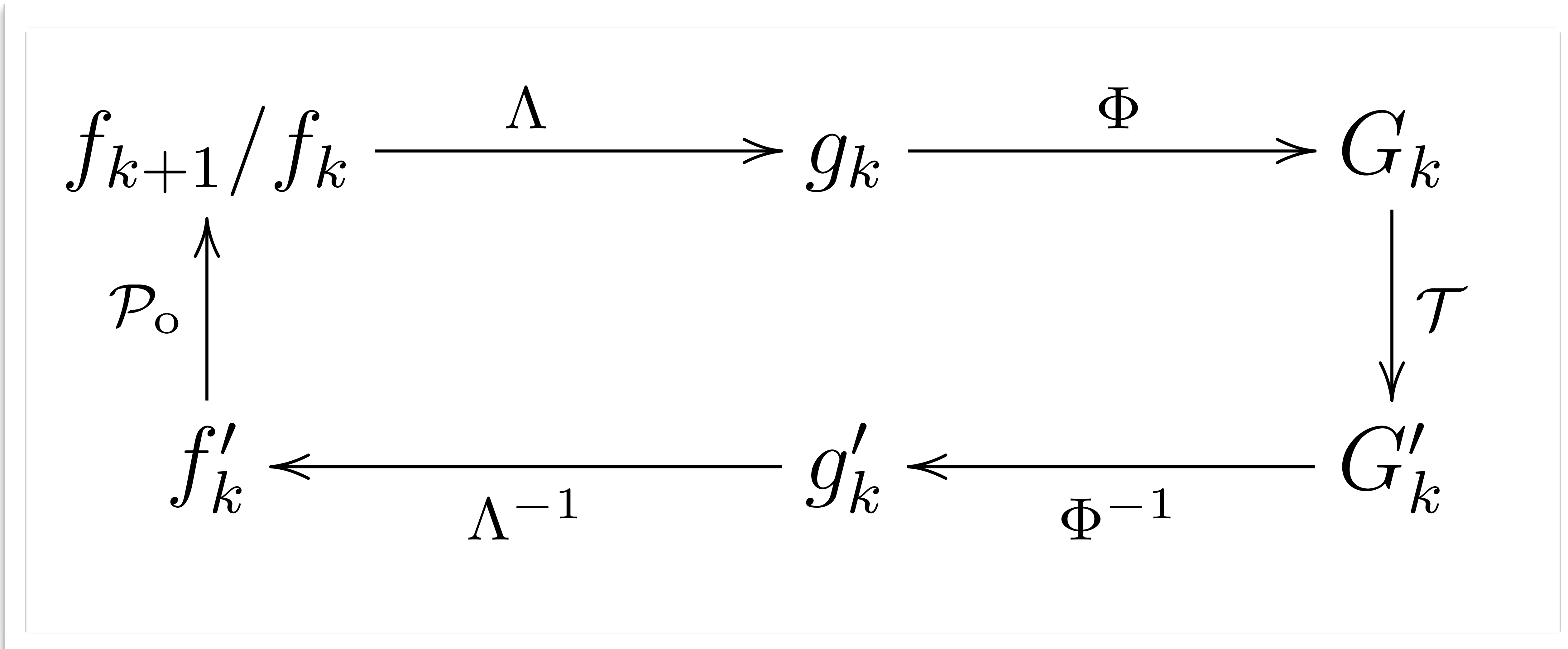
(i) $\{\lambda_2(\mathrm{n})\}$ are i.i.d. continuous random variables with respect to the Lebesgue measure on $\mathbb{S}^1$, $\mathbb{R}$ or $\mathbb{C}$ and $\{\lambda_2(\mathrm{n})\}$ are independent of $\{\lambda_1(\mathrm{n})\}$;

(ii) $\{\lambda_2(\mathrm{n})\}$ are deterministic;

then with probability one $f(\mathrm{n})$ is uniquely determined, up to a constant phase factor, by the Fourier magnitude measurements with two masks λ_1 and λ_2.

Alternating projections

Gerchberg-Saxton; Error Reduction (Fienup)



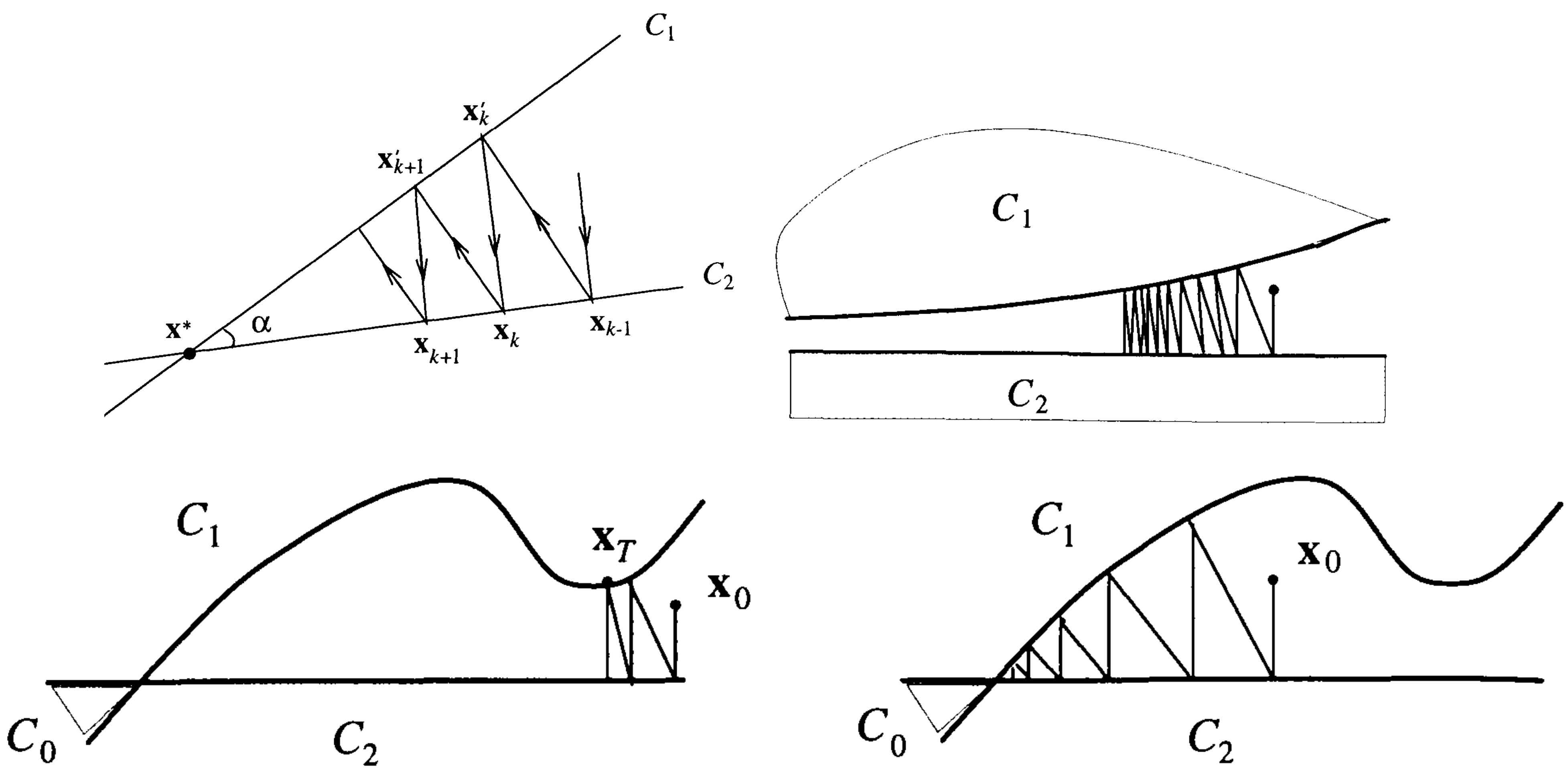
Multiple masks

$$\mathcal{P}_1 = \Lambda_1^{-1}\Phi^{-1}\mathcal{T}_1\Phi\Lambda_1$$

$$\mathcal{P}_2 = \Lambda_2^{-1}\Phi^{-1}\mathcal{T}_2\Phi\Lambda_2.$$

$$f_{k+1} = \mathcal{P}_o\mathcal{P}_2\mathcal{P}_1 f_k.$$

Error Reduction (Gerchberg-Saxton)



Bregman 65: convex constraints $\Longrightarrow$ convergence to a feasible solution.

Fourier magnitude data are a non-convex constraint!

Nonconvexity or nonuniqueness ?

Convergence

<u>THEOREM</u>

Let the object f be rank ≥ 2. Let h be a fixed point of $\mathcal{P}_{\mathrm{o}}\mathcal{P}_{\mathrm{f}}$ such that $\mathcal{P}_{\mathrm{f}}h$ satisfies the zero-padding condition.

(a) **If f is real-valued, $h = \pm f$ with probability one,**

(b) **If f satisfies the sector condition, then $h = e^{i\nu}f$, with probability at least**

$$1 - |\mathcal{N}| \left| \frac{b-a}{2\pi} \right|^{[S/2]}.$$

<u>LEMMA</u> Suppose the object constraint is convex. Then

$$\|\mathcal{P}_{\mathrm{f}}\{f_{k+1}\} - f_{k+1}\| \leq \|\mathcal{P}_{\mathrm{f}}\{f_k\} - f_k\|.$$

The equality holds if and only if $f_{k+1} = f_k$.

Error metrics

Relative error

$$e(\hat{f}) = \begin{cases} \|f - \hat{f}\| / \|f\| \\ \min\limits_{\nu \in [0, 2\pi)} \|f - e^{i\nu}\hat{f}\| / \|f\| \end{cases}$$

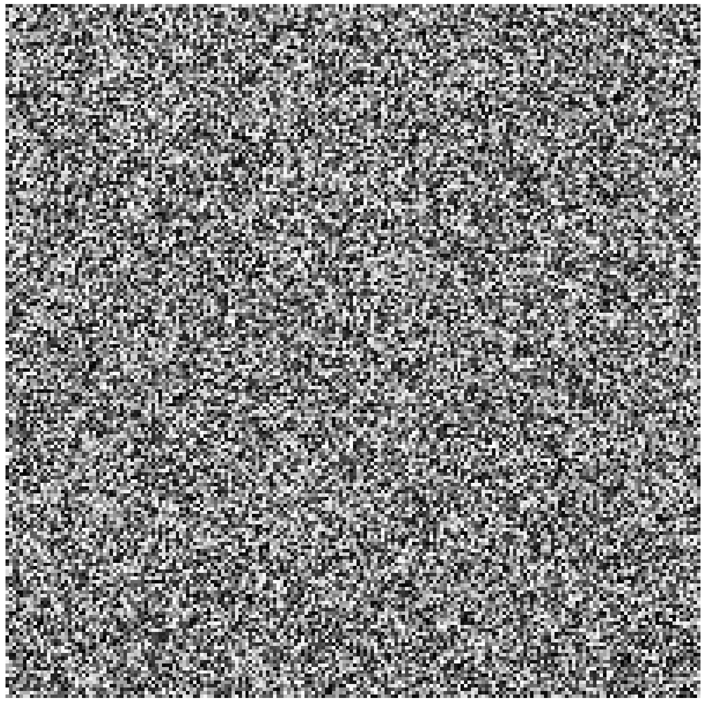
Relative residual

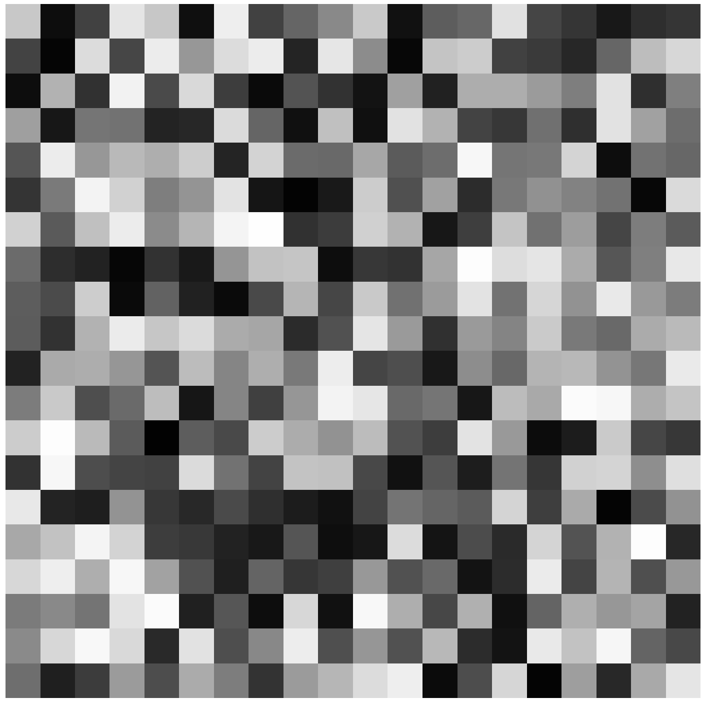
$$r(\hat{f}) = \frac{\| \, Y - |\Phi \Lambda \mathcal{P}_{\mathrm{o}}\{\hat{f}\}| \, \|}{\|Y\|}$$

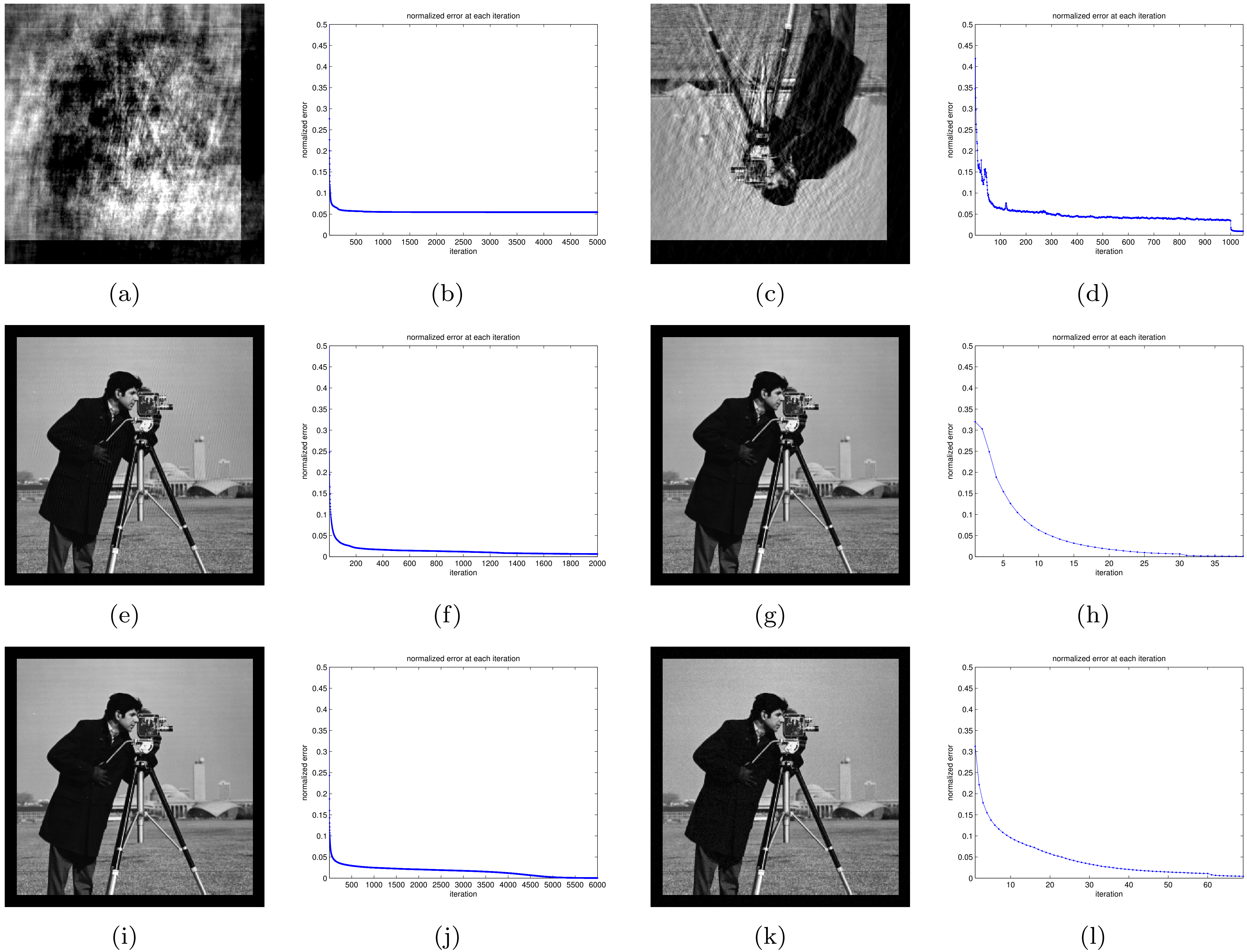
Objects with Loose Support

(a)

(b)

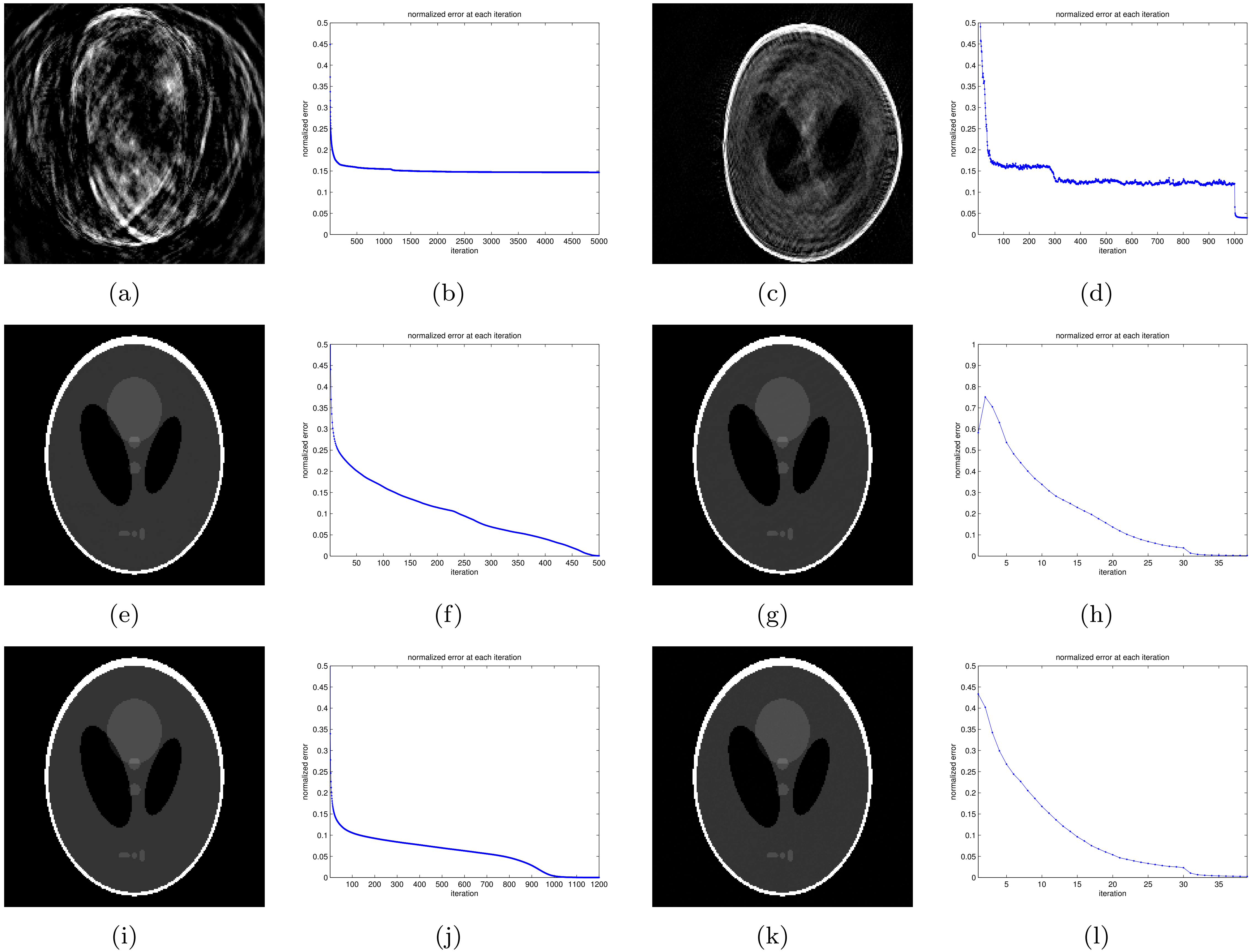






(e)-(h) Low resolution 40 x 40-block mask with OR=2

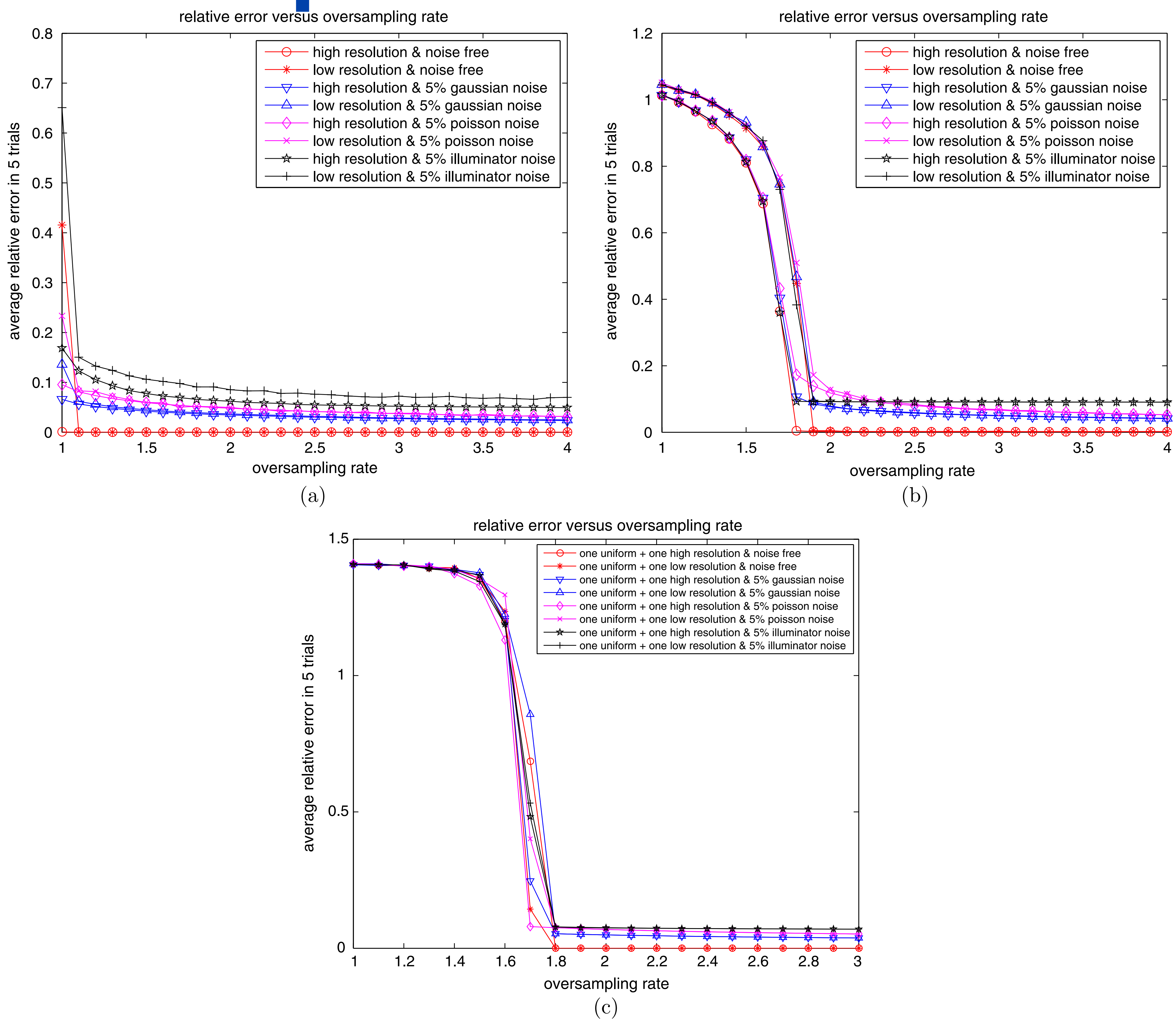
(i)-(l) High resolution mask with OR=1



(e) - (h) Low resolution 40 x 40-block mask with OR=2

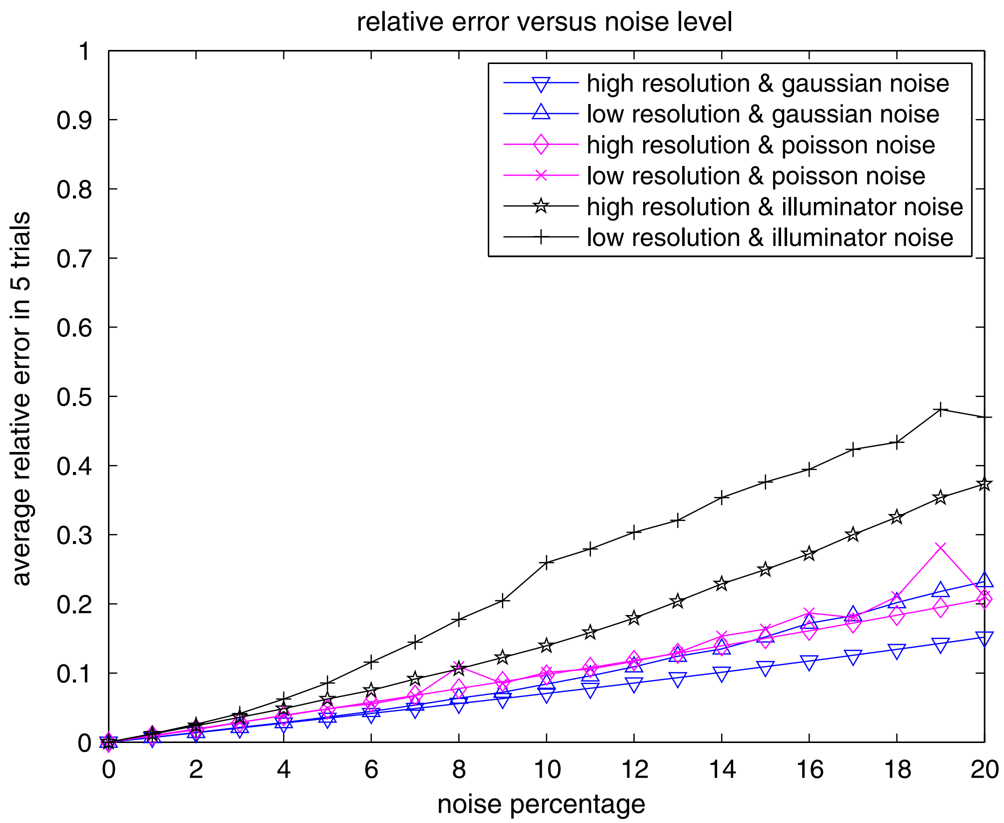
(i) - (l) High resolution mask with OR=1

Compressed measurement

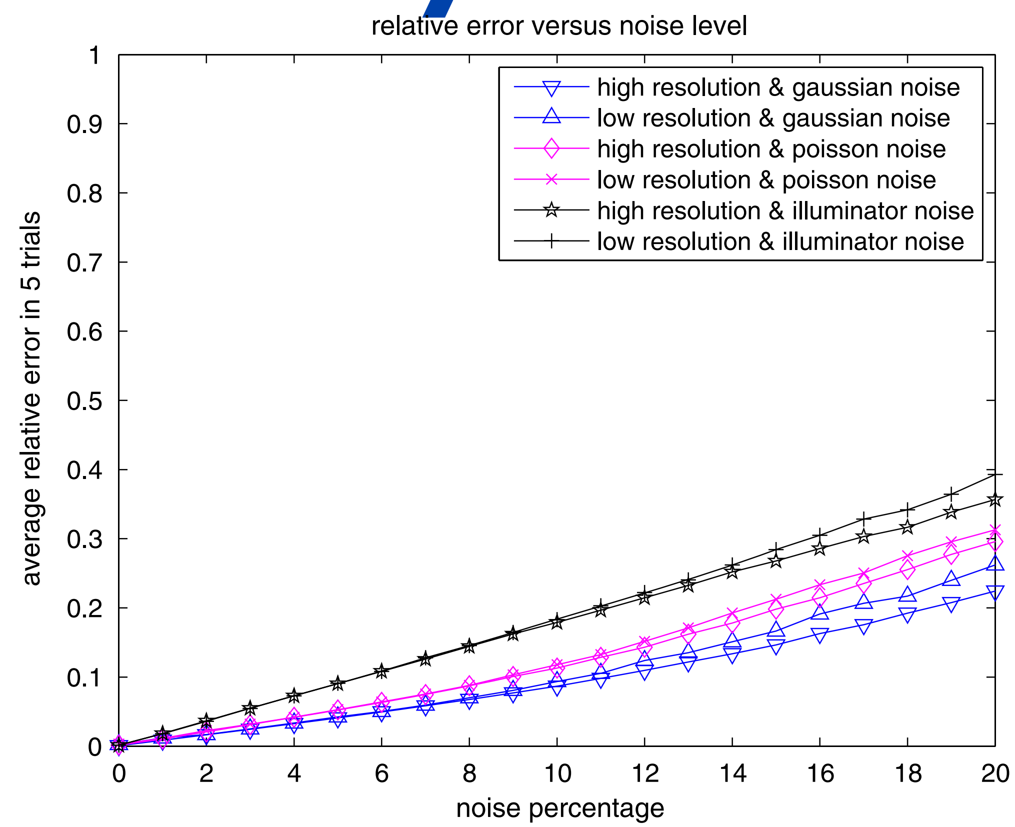


(a) real-valued (b) positive real & imaginary parts (c) no constraint

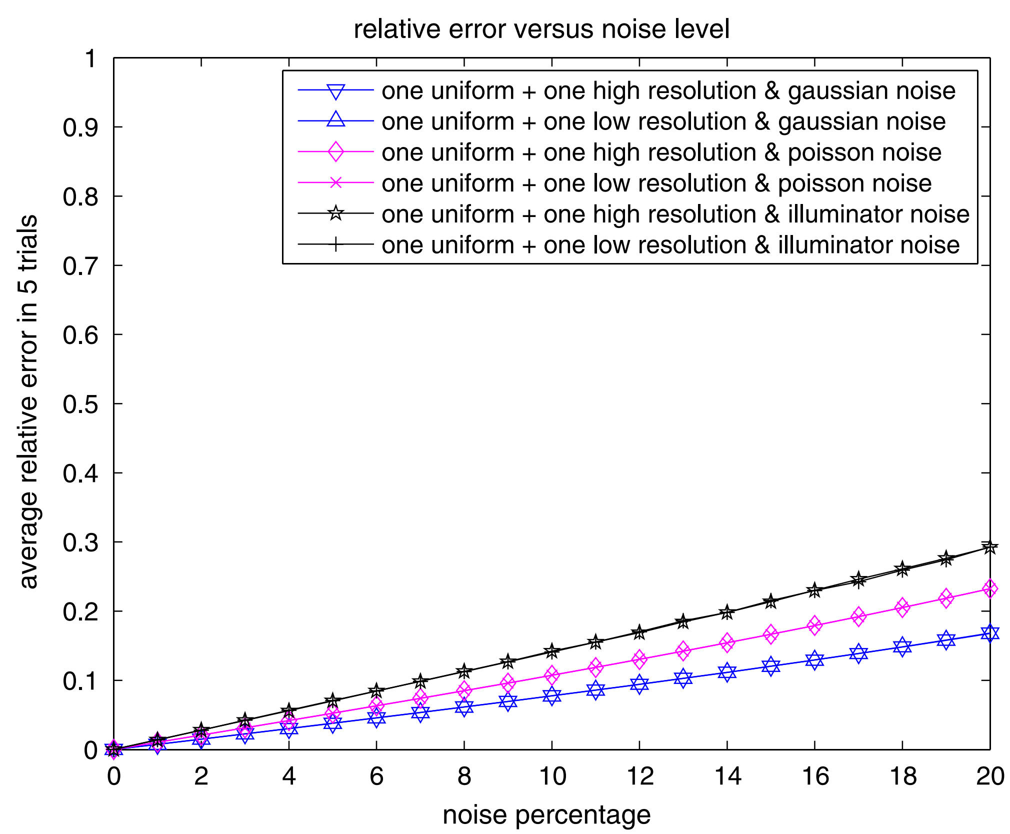
Noise stability



(a)



(b)



(c)

(a) real-valued objects (b) positive real & imaginary parts (c) no constraint

Roughly known mask

<u>THEOREM</u> Let the object f be positive and rank ≥ 2. Suppose the uncertainty of the mask estimate $\lambda_0 = \{\exp(i\phi_0(\mathbf{n}))\}$ is

$$\phi(\mathbf{n}) \in [\![\phi_0(\mathbf{n}) - \delta\pi, \phi_0(\mathbf{n}) + \delta\pi]\!]$$

with $\delta < 1$.

Suppose that another non-negative image $\tilde{f}$ and mask estimate $\tilde{\lambda} = \{\exp(i\tilde{\phi}(\mathbf{n}))\}$ satisfying

$$\tilde{\phi}(\mathbf{n}) \in [\![\phi_0(\mathbf{n}) - \delta\pi, \phi_0(\mathbf{n}) + \delta\pi]\!]$$

together produce the same Fourier intensity data on $\mathcal{L}$ as do f and λ. Then, with probability no less than

$$1 - |\mathcal{N}|\delta^{\lfloor S/2 \rfloor}$$

$\tilde{f}(\mathbf{n}) = f(\mathbf{n})\ \forall \mathbf{n}$ and furthermore $\tilde{\phi}(\mathbf{n}) = \theta + \phi(\mathbf{n})$ for a constant $\theta \in \mathbb{R}$ wherever $f(\mathbf{n}) \neq 0$.

Sketch of proof

There exist some $\mathbf{m}$ and $\theta \in [0, 2\pi]$ such that either

$$\tilde{\lambda}(\mathbf{n})\tilde{f}(\mathbf{n}) = \exp(i\theta)\lambda(\mathbf{m}+\mathbf{n})f(\mathbf{m}+\mathbf{n})$$

or

$$\tilde{\lambda}(\mathbf{n})\tilde{f}(\mathbf{n}) = \exp(i\theta)\overline{\lambda(\mathbf{m}-\mathbf{n})f(\mathbf{m}-\mathbf{n})}.$$

In the former case,

$$\tilde{f}(\mathbf{n}) = \exp(i\theta)\frac{|\lambda(\mathbf{m}+\mathbf{n})|\exp(i\phi(\mathbf{n}+\mathbf{m}))}{|\tilde{\lambda}(\mathbf{n})|\exp(i\measuredangle\tilde{\lambda}(\mathbf{n}))}f(\mathbf{n}+\mathbf{m})$$

Roughly known mask

<u>THEOREM</u> Let f be a complex-valued object of rank ≥ 2.

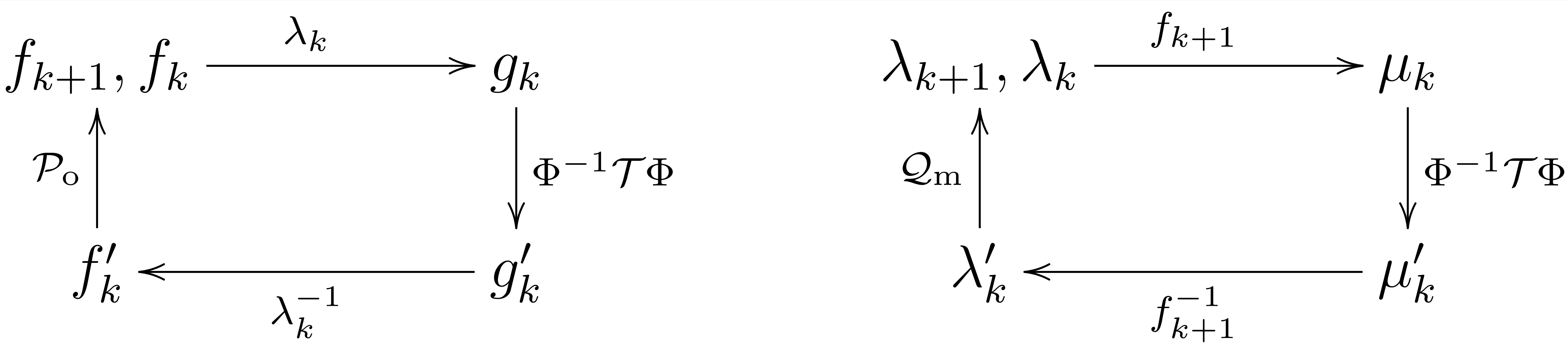
Let the first mask $\lambda^{(1)}$ is only roughly known with uncertainty δ. Suppose the second mask $\lambda^{(2)}$ is exactly known and assume the non-degeneracy condition on $\lambda^{(2)} f$.

Suppose that for a phase mask $\tilde{\lambda}$ of the same uncertainty δ and an object $\tilde{f}$ produce the same Fourier magnitudes on $\mathcal{L}$. Then with probability no less than

$$1 - |\mathcal{N}|\delta^{\lfloor S/2 \rfloor}$$

$\tilde{f}(\mathbf{n}) = \exp(i\nu_1) f(\mathbf{n}), \forall \mathbf{n}$, and $\tilde{\lambda}(\mathbf{n}) = \exp(i\nu_2)\lambda(\mathbf{n})$ if $f(\mathbf{n}) \neq 0$.

Object & Mask updates



(a) object update (b) mask update

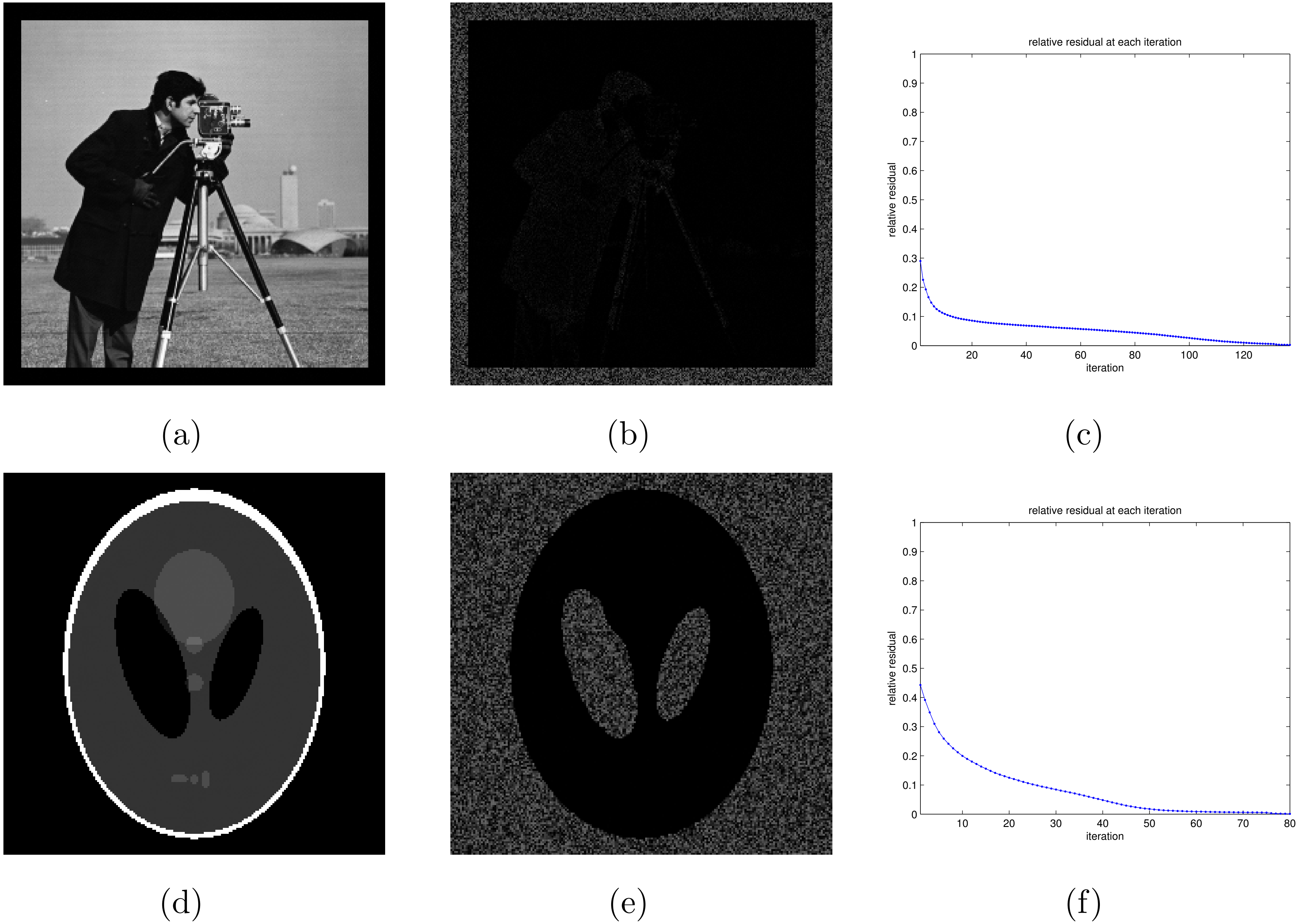


Fig. 5. Recovery of non-negative images with one LRM of $\delta = 0.2$. (a) the recovered cameraman $\hat{f}$ by 131 DRER + 6 AER steps. $e(\hat{f}) \approx 1.43\%$ and $\rho(\hat{f}, \hat{\lambda}) \approx 0.25\%$. (d) the recovered phantom $\hat{f}$ by 75 DRER + 5 AER steps. $e(\hat{f}) \approx 0.33\%$ and $\rho(\hat{f}, \hat{\lambda}) \approx 0.12\%$. The middle column shows the absolute phase differences between λ and $\hat{\lambda}$. The right column shows the relative residual at each iteration.

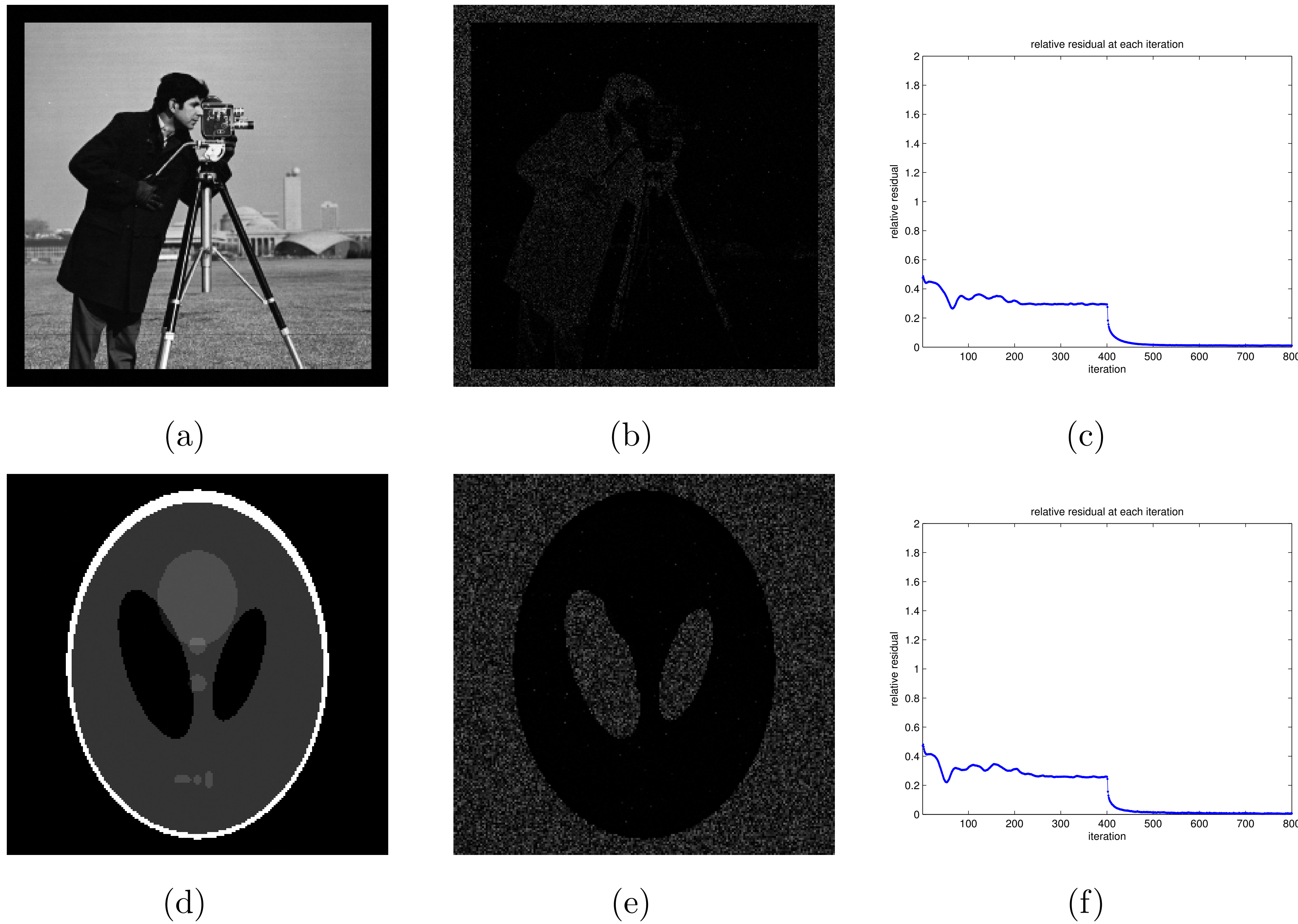
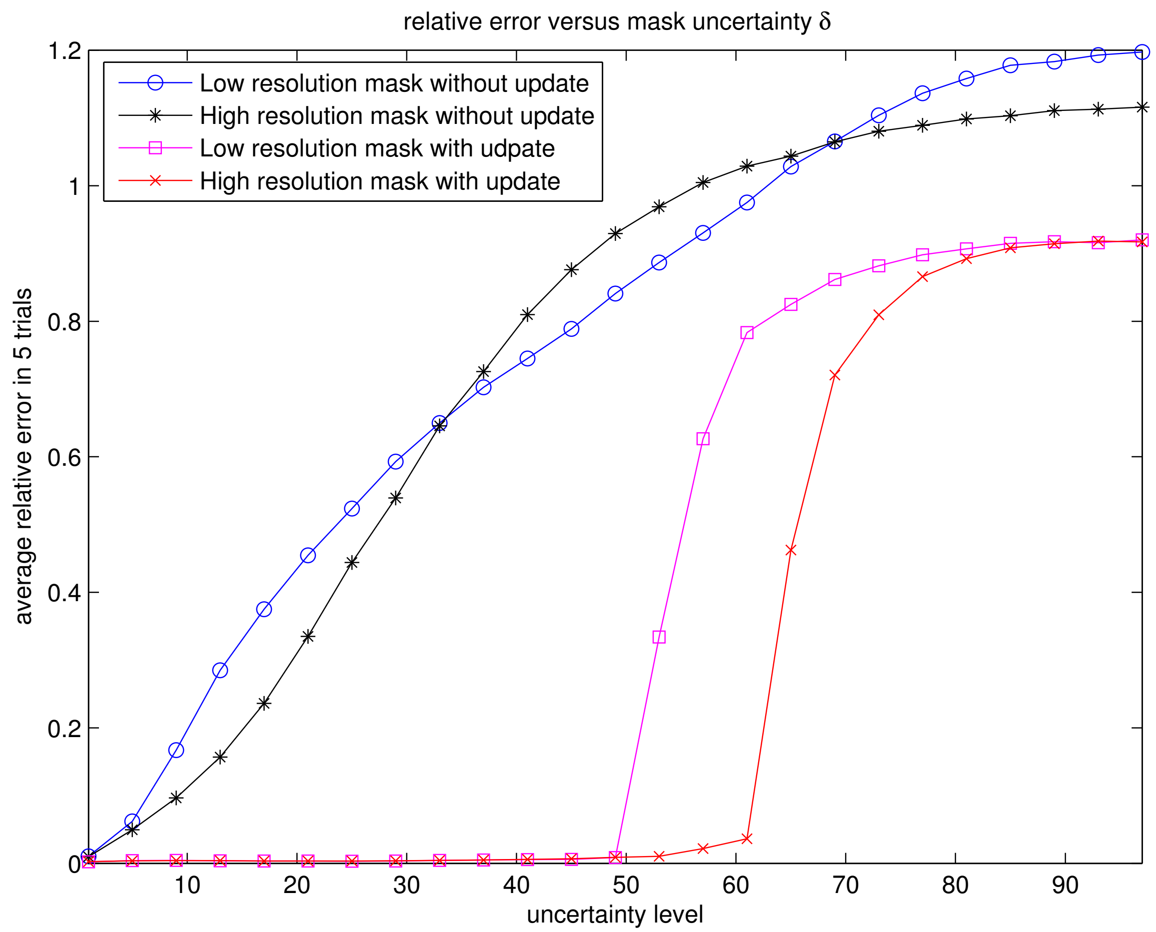
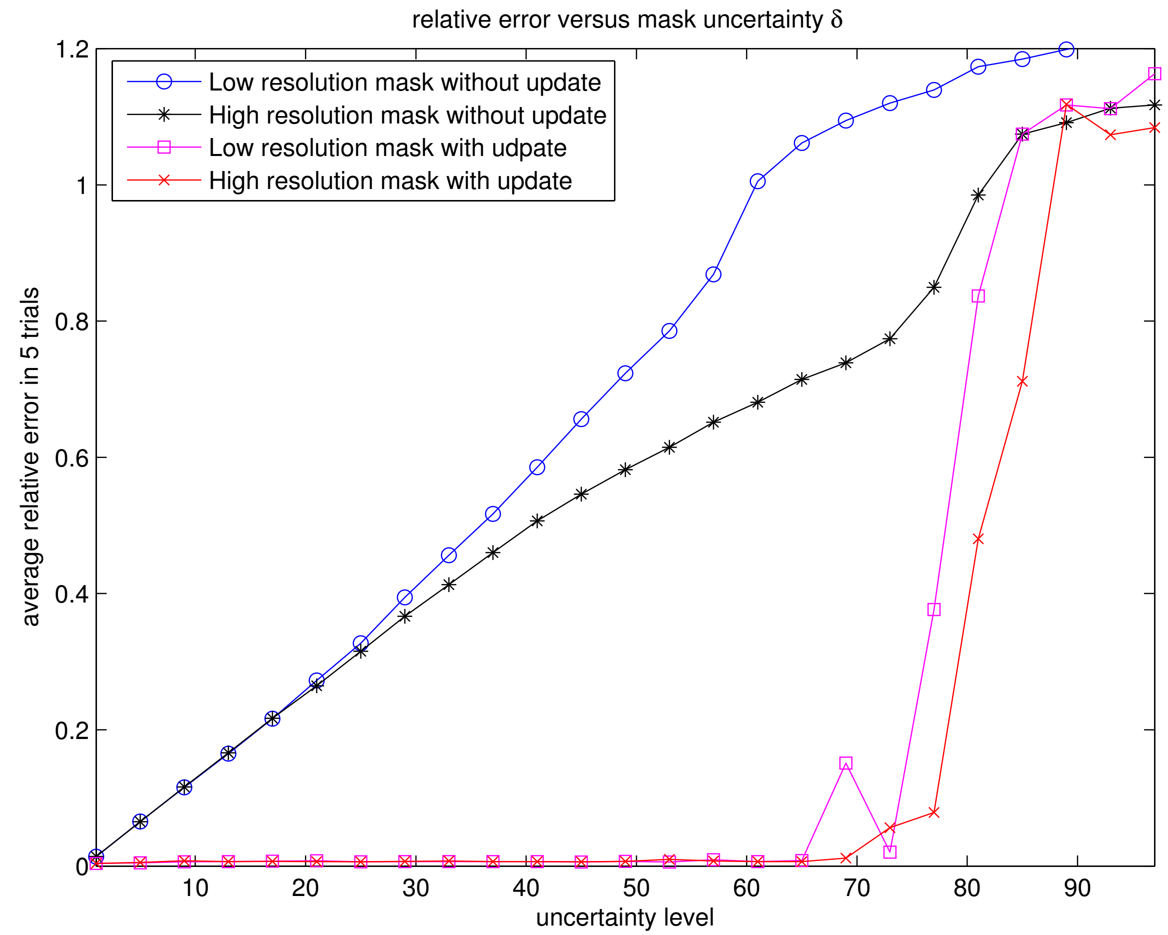
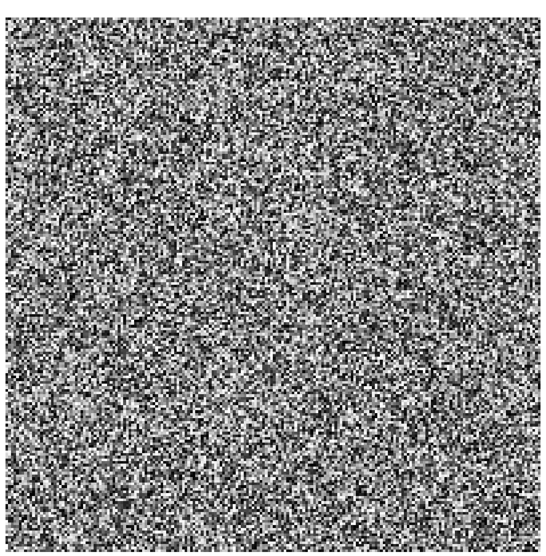
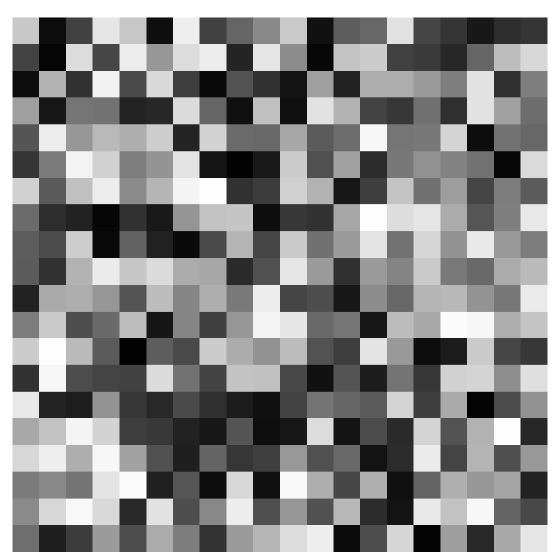


Fig. 6. Recovery of generic complex-valued images with one UM and one LRM of $\delta = 0.2$. (a) absolute values of the recovered cameraman $\hat{f}$ by 400 DRER + 400 AER steps. $e(\hat{f}) \approx 2.39\%$ and $\rho(\hat{f}, \hat{\lambda}) \approx 1.03\%$. (d) absolute values of the recovered phantom $\hat{f}$ by 400 DRER + 400 AER steps. $e(\hat{f}) \approx 1.37\%$ and $\rho(\hat{f}, \hat{\lambda}) \approx 0.65\%$. The middle column shows the absolute phase differences between λ and $\hat{\lambda}$. The right column shows the relative residual at each iteration.









Maximum of 200+1000 delta steps for DRER and AER separately

Conclusions

- Random mask as enabling tool for phase retrieval.
- Uniqueness
- Mask uncertainty
- Fast convergence
- OR =1 (real) or 2 (complex)
- References:

- F.: Inverse Problems 28 (2012) 075008
- F & Liao: Journal of Optical Society of America A 29 (2012), 1847-1859.
- F & Liao: Phase retrieval with roughly known mask, arXiv:1212.3858