The vector field of an autonomous system

We get a better geometric understanding of the solutions of a first-order system of differential equations if we rewrite the system as a vector equation that applies to a vector-valued function.

Consider the system

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

with independent variable $t$ and dependent variables $x$ and $y$. We use the right-hand side of this system to form a vector field

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

in the $xy$-plane. We also use $x(t)$ and $y(t)$ to form a vector-valued function

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then the (scalar) system of differential equations can be rewritten as one vector differential equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}).$$

**Example 1.** We consider the simple mass-spring system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -y.$$

First, let's rewrite this system in vector notation:

Consider the solution $(y_2(t), v_2(t)) = (\cos t, -\sin t)$ from last class. Let's express it in vector notation:

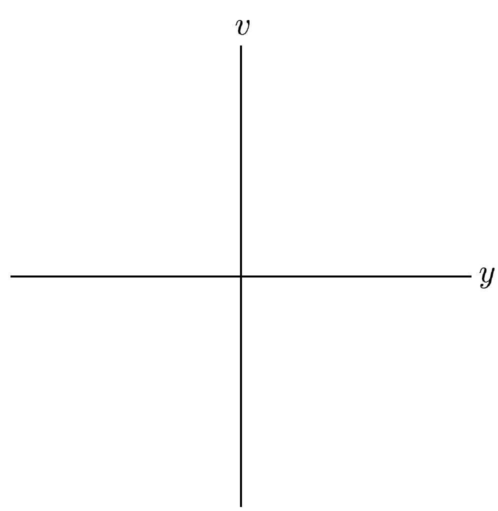
Now for the geometric interpretation of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}),$$

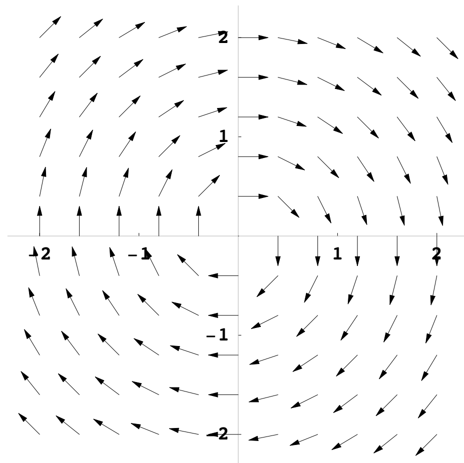
where

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -y \end{pmatrix}.$$

We use `HPGSystemSolver` to help visualize the vector field and the solutions.



The direction field associated with this system is




Here's another example:

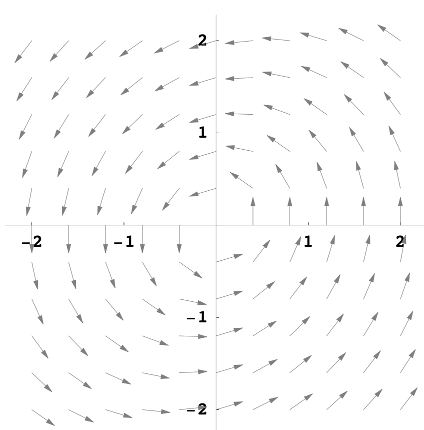
**Example 2.** Consider the system

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x - 0.3y.$$

The vector field associated with this system is

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x - 0.3y \end{pmatrix}.$$

Here's the direction field:




In this week's homework you have a matching problem in which you match systems of equations with their corresponding direction fields. There is another one of these problems in the old exams. Doing these problems is a good way to make sure that you understand how a system of differential equations determines a vector field.

Analytic techniques:

There are few analytic techniques that work for both linear and nonlinear systems.

1. You can always check to see if a given function is a solution (no wrong answers).

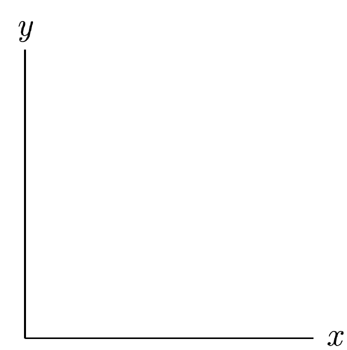
   For example, consider the initial-value problem

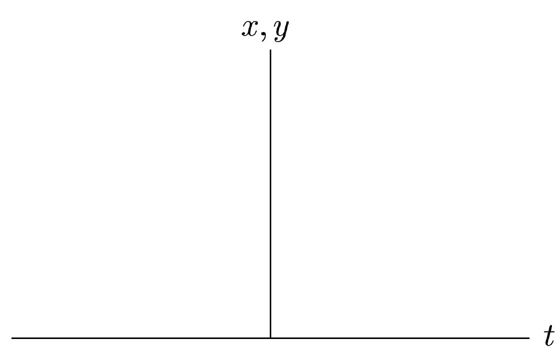
$$\begin{aligned} \frac{dx}{dt} &= 2y - x \\ \frac{dy}{dt} &= y \end{aligned} \qquad (x_0, y_0) = (2, 1).$$

   Using the vector notation $\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ we can write this initial-value problem as

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2y - x \\ y \end{pmatrix}, \qquad \mathbf{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

   First, let's see what the solution looks like when we graph it with `HPGSystemSolver`:



Claim: The function $\mathbf{Y}(t) = \begin{pmatrix} e^t + e^{-t} \\ e^t \end{pmatrix}$ solves the initial-value problem.