More on the consequences of uniqueness for autonomous systems

Recall the metaphor of the corn field.

Given the autonomous system $d\mathbf{Y}/dt = \mathbf{F}(\mathbf{Y})$. Let $\mathbf{Y}_0$ be an initial condition such that $\mathbf{Y}_1(t)$ is a solution that satisfies $\mathbf{Y}(t_1) = \mathbf{Y}_0$ and $\mathbf{Y}_2(t)$ is another solution that satisfies $\mathbf{Y}(t_2) = \mathbf{Y}_0$. Then

$$\mathbf{Y}_2(t) = \mathbf{Y}_1(t - (t_2 - t_1)).$$

Example. Consider the second-order equation $\dfrac{d^2y}{dt^2} + y = 0$ and its equivalent system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -y.$$

Note that

$$\mathbf{Y}_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

are both solutions to the system. How are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ related?

There is an animation on the web site that illustrates this phenomenon.

Here is an informal restatement of this consequence of uniqueness:

> For an autonomous system, if two solution curves in the phase plane touch, then they are identical.

Linear systems

Linear systems and second-order linear equations are the most important systems we study in this course.

What is a linear system with two dependent variables?

What is a second-order, homogeneous, linear equation?

Linear systems written in vector notation suggest the use of matrix multiplication:

Recall two examples that we have already discussed.

Example 1. We have already calculated the general solution to the partially decoupled system

$$\frac{dx}{dt} = 2y - x$$
$$\frac{dy}{dt} = y.$$

It is

$$x(t) = y_0 e^t + (x_0 - y_0)e^{-t}$$
$$y(t) = y_0 e^t.$$

Example 2. For the damped harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

and its equivalent system

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -2y - 3v,\end{aligned}$$

we used a guessing technique to find two (scalar) solutions $y_1(t) = e^{-t}$ and $y_2(t) = e^{-2t}$. In vector form, these solutions are written as

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \end{pmatrix} = e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Given a linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$, how do we calculate the vector in the vector field at any given point $\mathbf{Y}_0$?

How do we calculate the equilibrium points of $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$?

Example. Let $\mathbf{A}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

Example. Let $\mathbf{A}_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

Theorem. The origin is always an equilibrium point of a linear system. It is the only equilibrium point if and only if $\det \mathbf{A} \neq 0$.