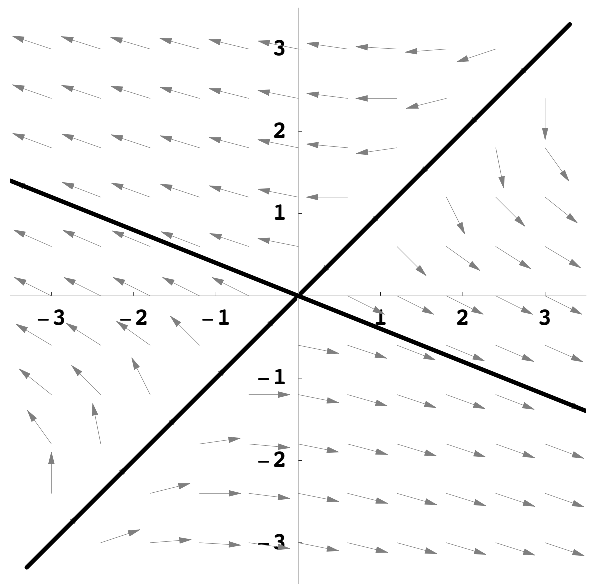
Eigenvalues, eigenvectors, general solutions, and phase portraits

To find the straight-line solutions of the linear system $d\mathbf{Y}/dt = \mathbf{AY}$, we need to find the eigenvalues and associated eigenvectors of the matrix $\mathbf{A}$. We find the eigenvalues using the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

**Example.** Find the general solution to $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix}\mathbf{Y}$.

Using `HPGSystemSolver`, we plot the phase portrait for this system.




Facts about eigenvalues and eigenvectors: Given a $2 \times 2$ matrix $\mathbf{A}$,

1. The characteristic equation can have two real roots, one real root of multiplicity two, or two complex conjugate roots.

2. Given an eigenvector $\mathbf{Y}_0$ associated to an eigenvalue $\lambda$, then any nonzero scalar multiple $\mathbf{Y}_0$ is also an eigenvector associated to $\lambda$.

3. Eigenvectors associated to distinct eigenvalues are linearly independent.

Summary of Case of Two Distinct Real Eigenvalues

Suppose $\mathbf{A}$ is a matrix with two eigenvalues $\lambda_1$ and $\lambda_2$. To be consistent, we will assume that $\lambda_1 < \lambda_2$, that $\mathbf{V}_1$ is an eigenvector associated to $\lambda_1$, and that $\mathbf{V}_2$ is an eigenvector associated to $\lambda_2$. The general solution of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

is $\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2$.

Case 1: $\lambda_1 < \lambda_2 < 0$.

**Example.** Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$