

---

---

Last class we studied an example with complex eigenvalues.

**Example.** Consider  $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$  where  $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$ .

Its eigenvalues are  $\lambda = -2 \pm i$ , and by solving the eigenvector equation  $\mathbf{A}\mathbf{Y}_0 = (-2 + i)\mathbf{Y}_0$ , we derived the “straight-line solution”

$$\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

There are lots of questions that come with this formula. First, what does the formula mean? Second, what good is it given that we are interested in real-valued solutions to our linear systems?

Once again Euler comes to the rescue: Remember the power series for the exponential function? It is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Let's use this series where  $x = bi$ .

---

---

We use Euler's formula

$$e^{bi} = \cos b + i \sin b$$

applied to the complex-valued function  $e^{(a+bi)t}$ .

But why does this help us solve our differential equation?

**Theorem.** Consider  $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$ , where  $\mathbf{A}$  is a matrix with real entries. If  $\mathbf{Y}_c(t)$  is a complex-valued solution, then both

$$\operatorname{Re}\mathbf{Y}_c(t) \quad \text{and} \quad \operatorname{Im}\mathbf{Y}_c(t)$$

are real-valued solutions, and they are linearly independent.

---

---

Now we can derive the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}$$

using the complex-valued solution  $\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$ .

---

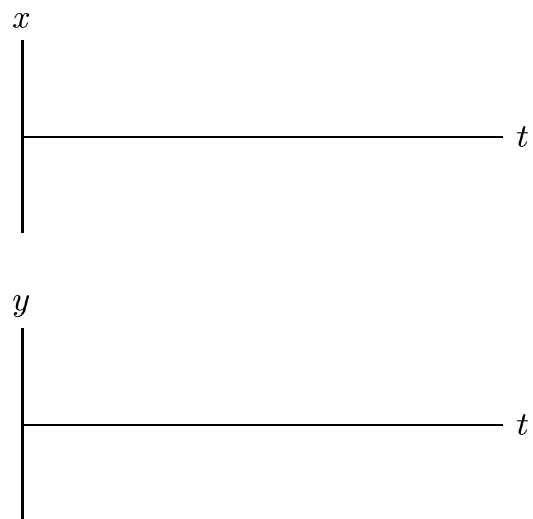
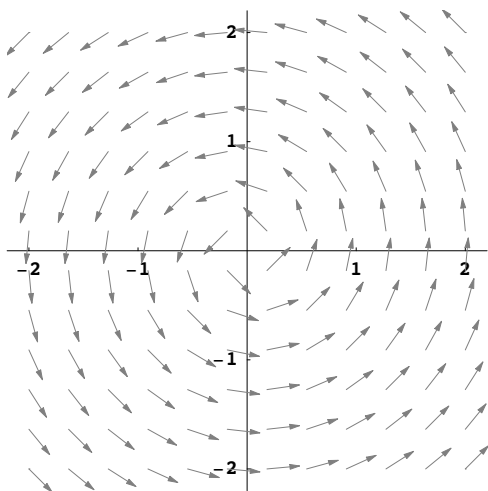
---

Three examples to illustrate the geometry of complex eigenvalues:

**Example 1.**  $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$  where  $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

The characteristic polynomial of  $\mathbf{A}$  is  $\lambda^2 + 1$ , so the eigenvalues are  $\lambda = \pm i$ . One eigenvector associated to the eigenvalue  $\lambda = i$  is

$$\mathbf{Y}_0 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$



---

---

**Example 2.**  $\frac{d\mathbf{Y}}{dt} = \mathbf{B}\mathbf{Y}$  where  $\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}$ .

The characteristic polynomial of  $\mathbf{B}$  is  $\lambda^2 + 4$ , so the eigenvalues are  $\lambda = \pm 2i$ . One eigenvector associated to the eigenvalue  $\lambda = 2i$  is

$$\mathbf{Y}_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$

