The geometry of complex eigenvalues

Example 1. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

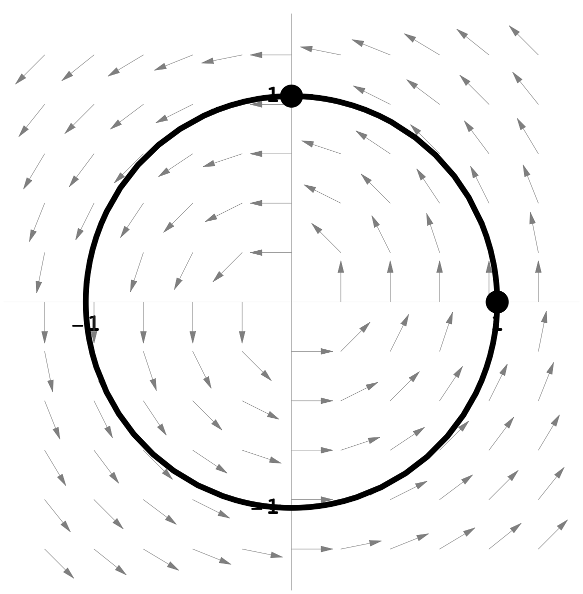
The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + 1$, so the eigenvalues are $\lambda = \pm i$. One eigenvector associated to the eigenvalue $\lambda = i$ is

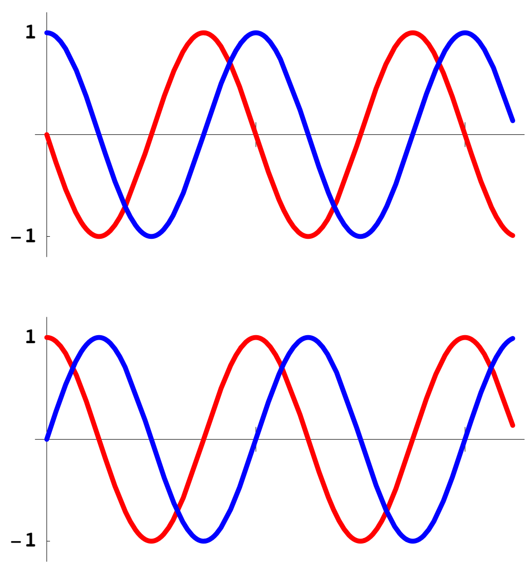
$$\mathbf{Y}_0 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

We obtain a general solution of the form

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + k_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

A solution curve and two pairs of $x(t)$- and $y(t)$-graphs are shown below.



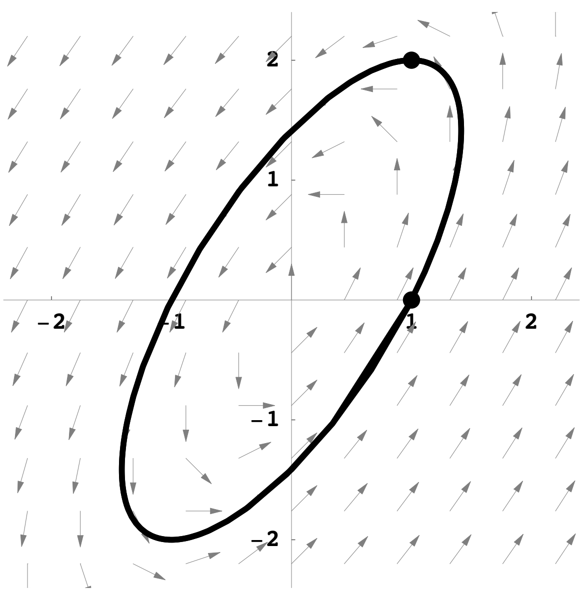


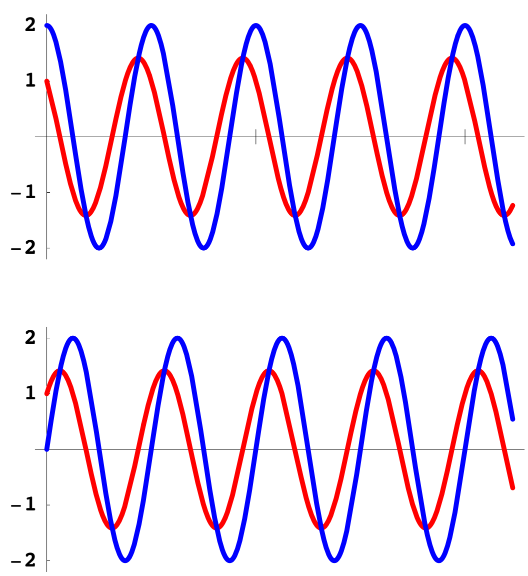
Example 2. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{BY}$ where $\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}$.

The characteristic polynomial of $\mathbf{B}$ is $\lambda^2 + 4$, so the eigenvalues are $\lambda = \pm 2i$. One eigenvector associated to the eigenvalue $\lambda = 2i$ is

$$\mathbf{Y}_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$

We get ellipses centered at the origin in the phase plane.

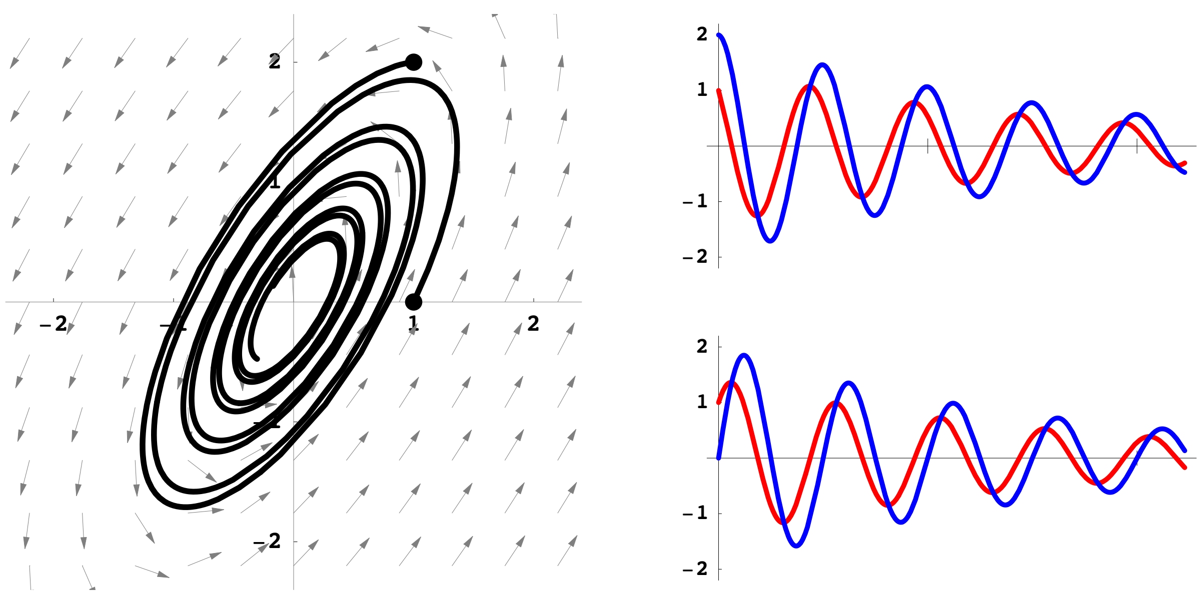




Example 3. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{C}\mathbf{Y}$ where $\mathbf{C} = \begin{pmatrix} 1.9 & -2 \\ 4 & -2.1 \end{pmatrix}$.

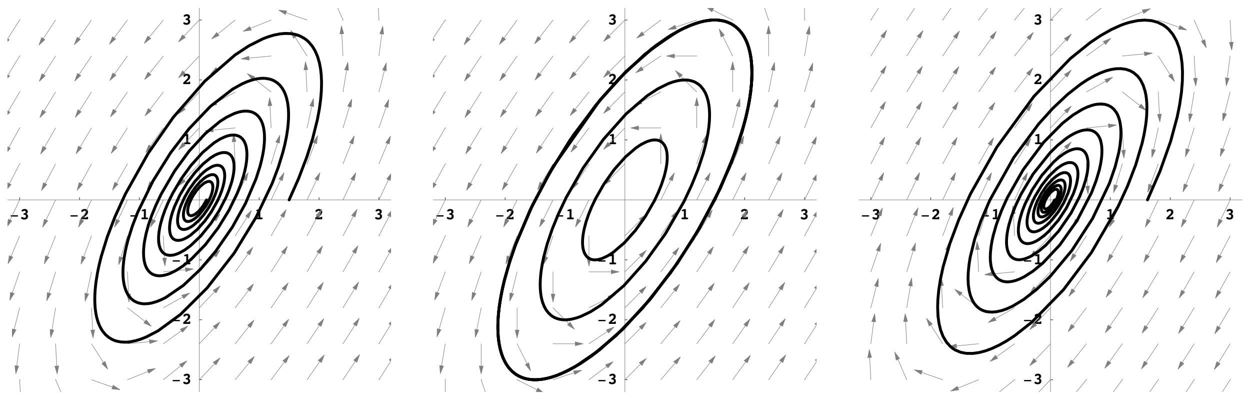
The characteristic polynomial of $\mathbf{C}$ is $\lambda^2 + 0.2\lambda + 4.01$, so the eigenvalues are $\lambda = -0.1 \pm 2i$. One eigenvector associated to the eigenvalue $\lambda = -0.1 + 2i$ is

$$\mathbf{Y}_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$



Summary: Linear systems with complex eigenvalues $\lambda = a \pm bi$

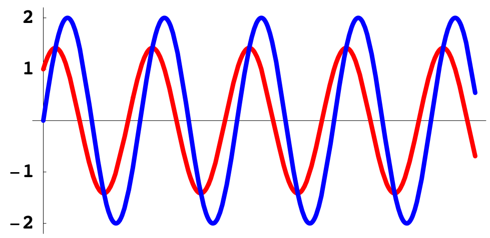
Here are the possible phase portraits:



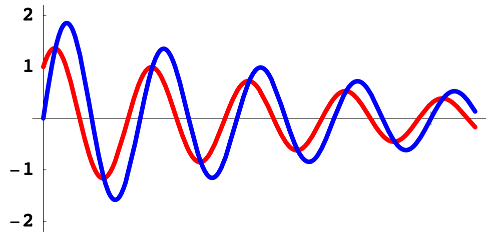
spiral sink ($a < 0$) center ($a = 0$) spiral source ($a > 0$)

What information can you get just from the complex eigenvalue alone?

Recall Example 2. The eigenvalues are $\lambda = \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



In Example 3, the eigenvalues are $\lambda = -0.1 \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



Frequency versus period: The solutions in Example 3 are not periodic in the strict sense. There is no time T such that

$$x(t+T) = x(t) \quad \text{and} \quad y(t+T) = y(t)$$

for all t. However, there is a period associated to these solutions. In the text, we call this the **natural period** of the solutions.

Perhaps it is best to think about these solutions as oscillating solutions that are decaying over time and to measure the oscillations in terms of their **frequency**.

Definition. The *frequency* F of an oscillating function $g(t)$ is the number of cycles that $g(t)$ makes in one unit of time.

Suppose that $g(t)$ is oscillating periodically with "period" T. What is its frequency F?

Example. Consider the standard sinusoidal functions $g(t) = \cos\beta t$ and $g(t) = \sin\beta t$.

Suppose we measure frequency in radians rather than in cycles. This measure of frequency is often called **angular frequency**. Let's denote the angular frequency by f. Then

$$f = 2\pi F.$$

Repeated eigenvalues

Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are "repeated."

Example. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$.

The characteristic polynomial of $\mathbf{A}$ is $(\lambda - 3)^2$, so there is only one eigenvalue, $\lambda = 3$. Let's calculate the associated eigenvectors:

But we already know how to solve this system. How?

We obtain the general solution $\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{3t} + 2y_0 t e^{3t} \\ y_0 e^{3t} \end{pmatrix}$.