Repeated eigenvalues

Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are "repeated."

Example. $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$.

The characteristic polynomial of $\mathbf{A}$ is $(\lambda - 3)^2$, so there is only one eigenvalue, $\lambda = 3$. Let's calculate the associated eigenvectors:

But we already know how to solve this system. How?

We obtain the general solution

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{3t} + 2y_0 t e^{3t} \\ y_0 e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{3t} \begin{pmatrix} 2y_0 \\ 0 \end{pmatrix}.$$

Note that this general solution is *not* written as a linear combination. Every nontrivial solution contains the first term, and most solutions contain both terms.

We use this result to motivate a different technique that we use to solve systems with repeated eigenvalues. We use a guessing technique where we guess a solution of the form

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + t e^{\lambda t}\mathbf{V}_1.$$

Note that the initial condition for this solution is $\mathbf{V}_0$.

Fact from linear algebra: If $\mathbf{A}$ is a 2×2 matrix with a repeated eigenvalue λ and $\mathbf{V}_0$ is any vector, then either

1. $(\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0 = \mathbf{0}$ (in other words, $\mathbf{V}_0$ is an eigenvector), or
2. the vector $\mathbf{V}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0$ is an eigenvector of $\mathbf{A}$.

Example. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$ where

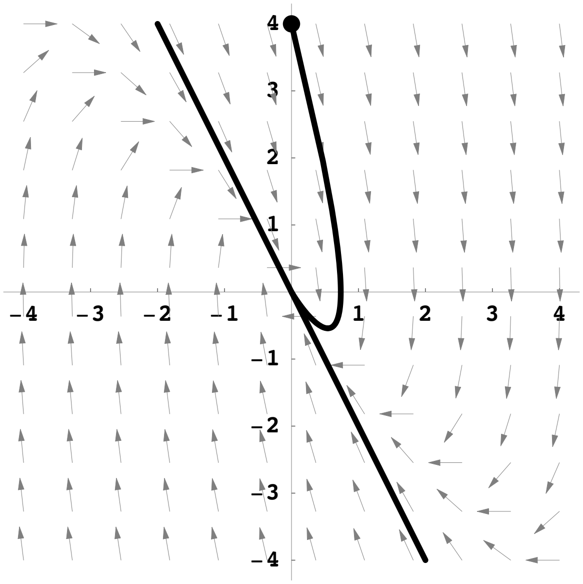
$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}.$$

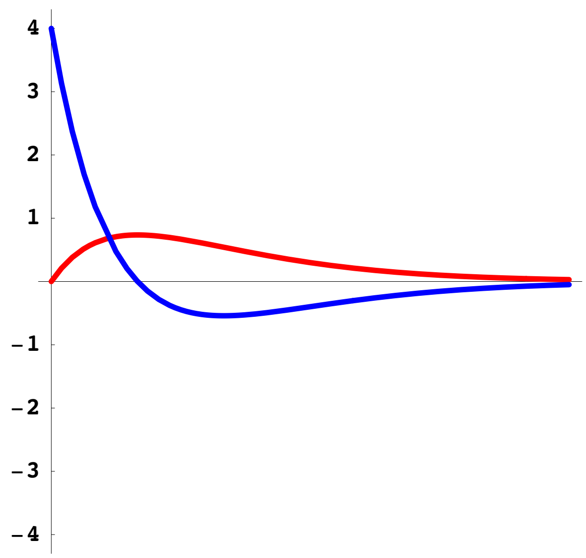
The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + 4\lambda + 4$, so $\lambda = -2$ is a repeated eigenvalue.

What is the long-term behavior of a system with a repeated, negative eigenvalue?

It is interesting to look at this example using two of the tools on the CD. Using `LinearPhasePortraits`, we can see that this system is on the boundary between spiral sinks and real sinks.

We can also use `HPGSystemSolver` to plot the phase portrait and a typical pair of $x(t)$- and $y(t)$-graphs.



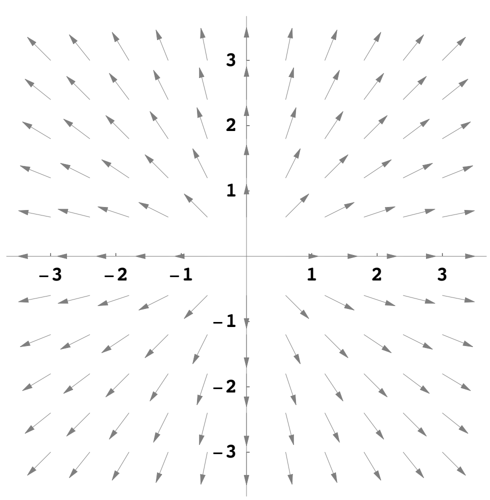


Unusual case of repeated eigenvalues: There is one type of linear system that has repeated eigenvalues that is different from the examples we have discussed.

Example. Consider $d\mathbf{Y}/dt = \mathbf{AY}$ where $\mathbf{A}$ is the diagonal matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

What are its eigenvalues and eigenvectors?



Finally consider the example

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \mathbf{Y}.$$

Its characteristic polynomial is $\lambda^2 + 3\lambda$. So its eigenvalues are $\lambda = -3$ and $\lambda = 0$. (If a system has 0 as an eigenvalue, we say that it is *degenerate*. The matrix $\mathbf{A}$ of coefficients is singular-