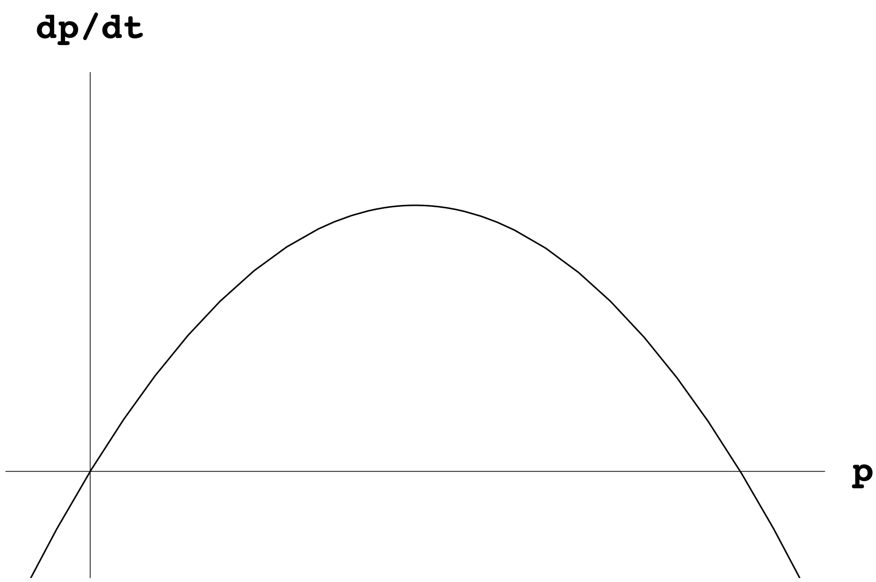
A Qualitative Analysis of the Logistic Model

Last Friday we derived the logistic model of population growth

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{N}\right).$$

Can we determine the long-term behavior of solutions without computing the solutions first?



A qualitative analysis of this equation yields the following observations about the solutions:

1. If $P_0 = 0$, then $dp/dt = 0$ for all t and therefore $p(t) = 0$ for all t.

2. If $P_0 = N$, then $dp/dt = 0$ for all t and therefore $p(t) = N$ for all t.

3. If $0 < P_0 < N$, then $dp/dt > 0$ for all t and therefore $p(t)$ is increasing for all t (need some theory we haven't studied yet).

4. If $P_0 > N$, then $dp/dt < 0$ for all t and therefore $p(t)$ is decreasing for all t (same issue regarding the theory).

A Numerical Simulation of the Logistic for the US Population

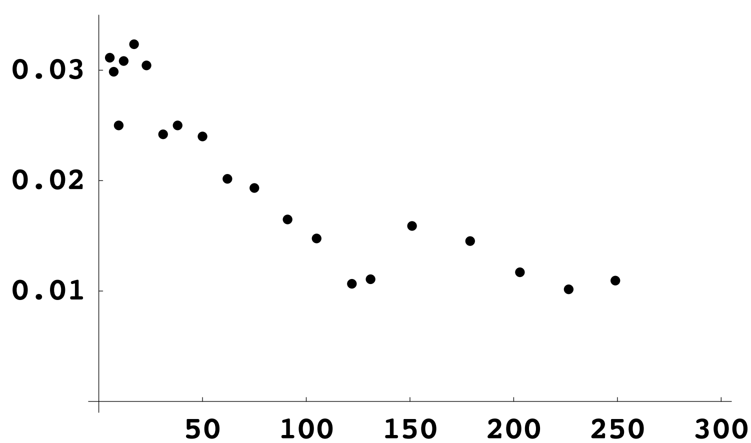
If we want to study this model numerically, we need estimates for k and N. How do we approximate the relative growth rates from the data?

Let's start by approximating the relative growth rate at 1800:

We can repeat this computation to produce approximate relative growth rates for 1800–1990:

Year	U.S. Population	Rel Growth Rate
1800	5.3	0.03113
1810	7.2	0.02986
1820	9.6	0.02500
1830	12	0.03083
1840	17	0.03235
1850	23	0.03043
1860	31	0.02419
1870	38	0.02500
1880	50	0.02400
1890	62	0.02016
1900	75	0.01933
1910	91	0.01648
1920	105	0.01476
1930	122	0.01066
1940	131	0.01107
1950	151	0.01589
1960	179	0.01453
1970	203	0.01170
1980	226	0.01015
1990	249	0.01094

Here's a graph of these relative growth rates versus population:

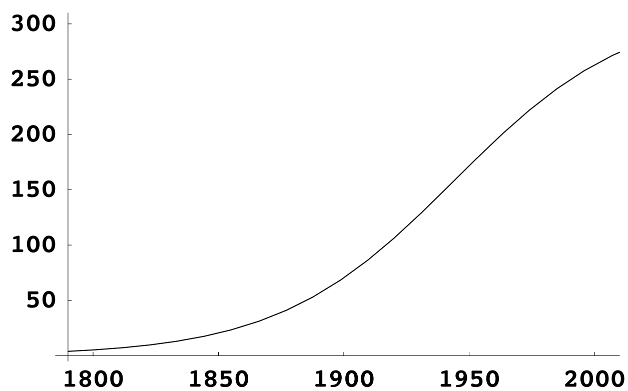


Using this statistical analysis, we obtain the differential equation

$$\frac{dp}{dt} = p(0.02846 - 0.00009p).$$

Assuming these numbers, what are the values of k and N?

Now we plot an approximate solution to this logistic differential equation with the initial condition $p(0) = 3.9$.



This completes our introduction to modeling via differential equations. We studied two models—the Malthusian model (exponential growth) and the logistic model, and three techniques were introduced:

1. an analytic technique to find the solutions of the Malthusian model
2. a qualitative technique to analyze the long-term behavior of solutions to logistic models
3. numerical techniques to approximate a solution to the logistic given by the U.S. population data.

General observations

Before we start discussing some of the basic techniques for studying differential equations, I want to make a few general observations about first-order differential equations

$$\frac{dy}{dt} = f(t, y)$$

and their solutions.

1. What is a differential equation and what is a solution to an **initial-value** problem?

2. Be careful about notation.

3. What does the term **general solution** mean?