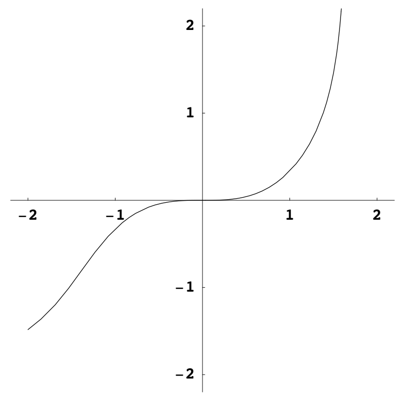
Euler's method

the initial-value problem

$$\frac{dy}{dt} = y^3 + t^2, \quad y(0) = 0.$$

Here's the graph of its solution.



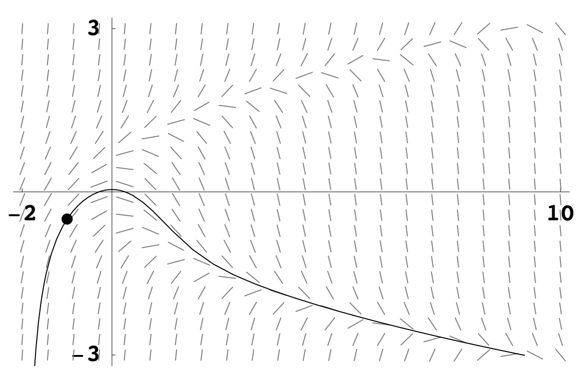
There isn't a nice formula for this function, so how do we obtain its graph? The answer is that we use a numerical algorithm to obtain an approximate solution.

Today we study the numerical algorithm known as Euler's method. It is the most basic of all of the numerical algorithms that are used to approximate solutions to differential equations. Let's start with an example to get an idea of how the method works.

Example. Consider the initial-value problem

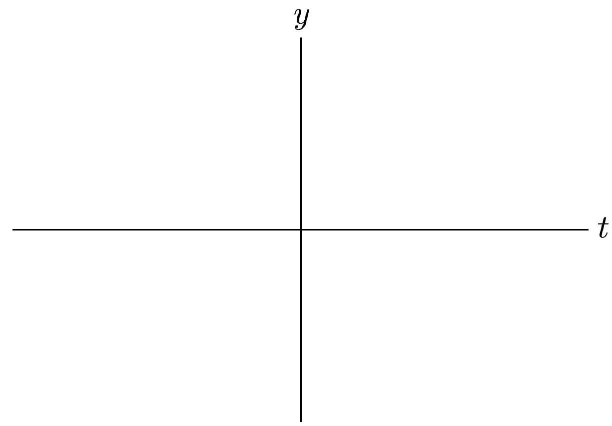
$$\frac{dy}{dt} = y^2 - t, \quad y(-1) = -\tfrac{1}{2}.$$

First, let's see what `HPGSolver` produces:



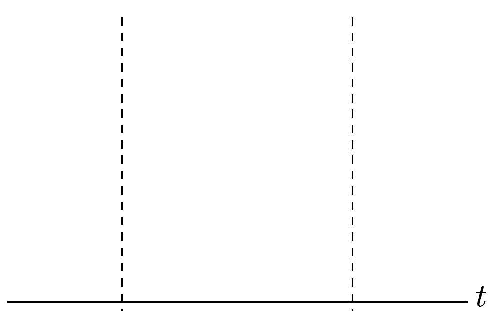
Now let's see what happens when we use Euler's method to approximate the solution with a step size of $\Delta t = 0.5$. We'll use the `EulersMethod` tool from `DETools`.

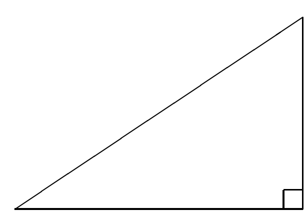
k	t_k	y_k	m_k
0	-1	-0.50	
1			
2			
3			
4			
5			
6			



Here is a general picture of the algorithm and the associated notation:

Let's look more closely at the kth step and the key triangle:





These observations yield Euler's method:

Euler's method is easy to program—even with just a spreadsheet.

Euler's method is the most basic "fixed-step-size" algorithm for numerically approximating solutions. `HPGSolver` also uses a fixed-step-size algorithm called the Runge-Kutta method. The Runge-Kutta method is usually more efficient and more accurate than Euler's method (see Chapter 7 of our text). Unfortunately, there are differential equations that are not amenable to fixed-step-size algorithms.

Example. Consider the initial-value problem

$$\frac{dy}{dt} = e^t \sin y, \quad y(0) = 5.$$

Let's see what happens when we use Euler's method to approximate the solution with various step sizes $0.01 \leq \Delta t \leq 0.1$.

The spreadsheet for this example is also posted on the course web site.