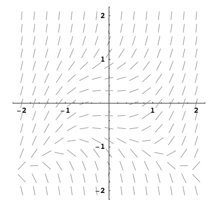
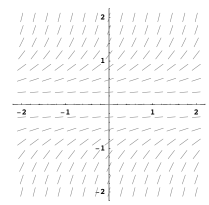
Existence and Uniqueness Theory

First we consider three examples to illustrate the idea of the domain of a differential equation:

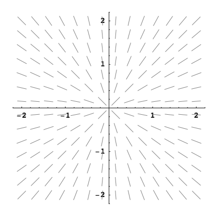
Example 1. $\dfrac{dy}{dt} = y^3 + t^2$



Example 2. $\dfrac{dy}{dt} = y^2$



Example 3. $\dfrac{dy}{dt} = \dfrac{y}{t}$



We start our discussion of the theory with the Existence Theorem:

Existence Theorem Suppose $f(t, y)$ is a continuous function in a rectangle of the form

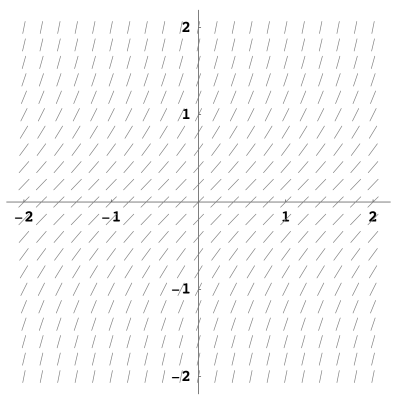
$$\{(t, y) \,|\, a < t < b,\ c < y < d\}$$

in the ty-plane. If (t_0, y_0) is a point in this rectangle, then there exists an $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ that solves the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \quad \blacksquare$$

What's the significance of the ϵ in the Existence Theorem?

Example. $\dfrac{dy}{dt} = 1 + y^2, \quad y(0) = 0$



What does the Existence Theorem tell us about the initial-value problem

$$\frac{dy}{dt} = y^3 + t^2, \quad y(0) = 0?$$

The other main theoretical result in differential equations is the Uniqueness Theorem.

Uniqueness Theorem Suppose $f(t, y)$ and $\partial f/\partial y$ are continuous functions in a rectangle of the form

$$\{(t, y) \mid a < t < b,\ c < y < d\}$$

in the ty-plane. If (t_0, y_0) is a point in this rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

for all t in the interval $t_0 - \epsilon < t < t_0 + \epsilon$ (where ϵ is some positive number), then

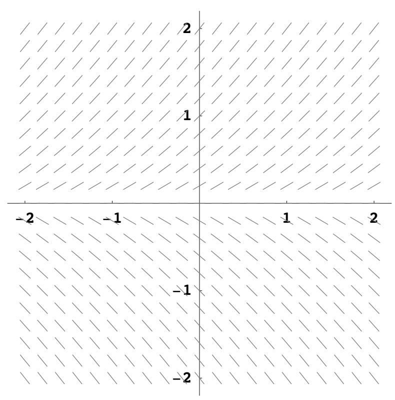
$$y_1(t) = y_2(t)$$

for $t_0 - \epsilon < t < t_0 + \epsilon$. That is, the solution to the initial-value problem is *unique*. ■

Here's an example that lacks uniqueness:

Example. $\dfrac{dy}{dt} = \sqrt[3]{y}$

(More blank space and the slope field for $dy/dt = \sqrt[3]{y}$ on the top of the next page.)



Bogus Example. The example

$$\frac{dy}{dt} = \frac{y}{t} + t\cos t$$

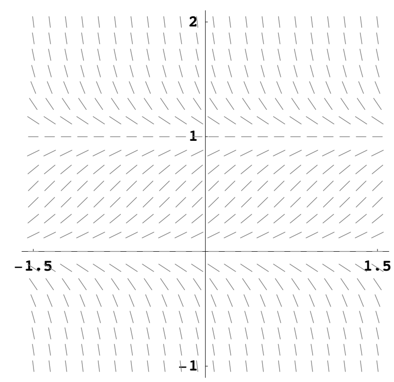
in `FirstOrderSystems` seems to violate the Uniqueness Theorem, but in fact it does not. Why?

More on Uniqueness

The Uniqueness Theorem has many useful consequences. Here are three examples:

Example 1. $\dfrac{dy}{dt} = -2ty^2$

Example 2. $\dfrac{dy}{dt} = 4y(1-y)$



Example 3. $\dfrac{dy}{dt} = e^t \sin y$