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Math 245

Homework 1

Due on January 17, 2025

The six problems below are all from the 2nd edition of EC1. Only turn in your **best two problems**.

Note that unless otherwise specified, you should always assume that variables like m, n are chosen from $\mathbb{N}$.

1.6 [3-] For $n \in \mathbb{Z}$, let

$$J_n(2x) = \sum_{k \in \mathbb{Z}} \frac{(-1)^k x^{n+2k}}{k!(n+k)!},$$

where we set $1/j! = 0$ for $j < 0$. Show that

$$e^x = \sum_{n \geq 0} L_n J_n(2x),$$

where $L_0 = 1$, $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$ for $n \geq 2$. (The numbers L_n for $n \geq 1$ are *Lucas numbers.*)

1.12 [2+] Choose n points on the circumference of a circle in "general position". Draw all $\binom{n}{2}$ chords connecting two of the points. ("General position" means that no three of these chords intersect in a point.) Into how many regions will the interior of the circle be divided? Give an elegant proof without using induction, finite differences, generating functions, summations, and the like.

(Previously, I had students turning in solutions involving Euler's formula. This should be avoided as well.)

1.26 [2] Let $\bar{c}(m, n)$ denote the number of compositions of n with largest part at most m. Show that

$$\sum_{n \geq 0} \bar{c}(m, n) x^n = \frac{1 - x}{1 - 2x + x^{m+1}}.$$

1.28 [2] Let $\kappa(n, j, k)$ be the number of weak compositions of n into k parts, each part less than j. Give a generating function proof that

$$\kappa(n, j, k) = \sum_{r+sj=n} (-1)^s \binom{k+r-1}{r} \binom{k}{s},$$

where the sum is over all pairs $(r, s) \in \mathbb{N}^2$ satisfying $r + sj = n$.

1.33 [2+]

a. Let $k, n \geq 1$. Find the number of sequences $\emptyset = S_0, S_1, \ldots, S_k$ of subsets of $[n]$ if for all $1 \leq i \leq k$ we have either (i) $S_{i-1} \subset S_i$ and $|S_i - S_{i-1}| = 1$, or (ii) $S_i \subset S_{i-1}$ and $|S_{i-1} - S_i| = 1$.

b. Suppose that we add the additional condition that $S_k = \emptyset$. Show that now the number $f_k(n)$ of sequences is given by

$$f_k(n) = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (n-2i)^k.$$

Note that $f_k(n) = 0$ if k is odd.

1.54 [2] How many n-element multisets on $[2m]$ are there satisfying: (i) $1, 2, ..., m$ appear at most once each, and (ii) $m+1, m+2, ..., 2m$ appear an even number of times each?

Your answer is supposed to be a closed formula instead of a summation.