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Math 245

Homework 2

Due on Jan 27, 2025

The seven problems below are all from the 2nd edition of EC1. Note that I treat 1.44a and 1.44b as two separate problems. However, clearly you should not use your solutions to 1.44b as your solutions to 1.44a.

Only turn in your **best two problems**.

You may turn in one problem rated as [2+] or [3-] from Homework 1 (which you have not previously submitted), but at least one problem has to be from this set.

1.38 [2] Show that the number of permutations $\omega \in \mathfrak{S}_n$ fixed by the fundamental transformation $\mathfrak{S}_n \xrightarrow{\wedge} \mathfrak{S}_n$ of Proposition 1.3.1 (i.e., $\omega = \hat{\omega}$) is the Fibonacci number F_{n+1}.

1.44a [2] Suppose $n \geq 2$. Show that the total number of cycles of all even permutations of $[n]$ and the total number of cycles of all odd permutations of $[n]$ differ by $(-1)^n(n-2)!$.

1.44b [3-] Give a bijective proof for the above problem.

1.53a [2] The *Eulerian Catalan number* is defined by $EC_n = A(2n+1, n+1)/(n+1)$. The first few Eulerian Catalan numbers, beginning with $EC_0 = 1$, are $1, 2, 22, 604, 31238$. Show that $EC_n = 2A(2n, n+1)$, whence $EC_n \in \mathbb{Z}$.

1.53b [3-] Show that EC_n is the number of permutations $\omega = a_1 a_2 \cdots a_{2n+1}$ with n descents, such that every left factor $a_1 a_2 \cdots a_i$ has at least as many ascents as descents. For $n = 1$ we are counting the two permutations 132 and 231.

1.126 [2+] Show that

$$\sum_{\omega} q^{\mathrm{inv}(\omega)} = q^n \prod_{j=0}^{n-1} (1 + q^2 + q^4 + \cdots + q^{4j}),$$

where ω ranges over all fixed-point free involutions in $\mathfrak{S}_{2n}$, and where $\mathrm{inv}(\omega)$ denotes the number of inversions of ω. Given a simple combinatorial proof analogous to the proof of Corollary 1.3.13.

1.138 [2] Find a simple formula for the number of alternating permutations $a_1 a_2 \dots a_{2n} \in \mathfrak{S}_{sn}$ satisfying $a_2 < a_4 < a_6 < \cdots < a_{2n}$.