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Math 245

Homework 3

Due on February 5, 2025

The six problems below are all from the 2nd edition of EC1. Note that 1.102ab is treated as one problem. Only turn in your **best two problems**.

You may turn in one problem rated as [2+] or [3-] from Homework 2, but at least one problem has to be from this set.

1.84 [2] Show that the number of partitions of n in which each part appears exactly 2, 3, or 5 times is equal to the number of partitions of n into parts congruent to ± 2, ± 3, $6 \pmod{12}$.

1.85 [2+] Prove that the number of partitions of n in which no part appears exactly once equals the number of partitions of n into parts not congruent to $\pm 1 \pmod 6$.

1.87 [3-] Let $A_k(n)$ be the number of partitions of n into odd parts (repetition allowed) such that exactly k distinct parts occur. For instance, when $n = 35$ and $k = 3$, one of the partitions being enumerated is $(9, 9, 5, 3, 3, 3, 3)$. Let $B_k(n)$ be the number of partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n such that the sequence $\lambda_1, \ldots, \lambda_r$ is composed of exactly k noncontiguous sequences of one or more consecutive integers. For instance, when $n = 44$ and $k = 3$, one of the partitions being enumerated is $(10, 9, 8, 7, 5, 3, 2)$, which is composed of $10, 9, 8, 7$ and 5 and $3, 2$. Show that $A_k(n) = B_k(n)$ for all k and n. Note that summing over all k gives Proposition 1.8.5, i.e., $p_{\text{odd}}(n) = q(n)$.

1.102ab [2]

a. Let x and y be variables satisfying the commutation relation $yx = qxy$, where q commutes with x and y. Show that

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

b. Generalize to $(x_1 + x_2 + \cdots + x_m)^n$, where $x_i x_j = q x_j x_i$ for $i > j$. Avoid using inductive or recursive arguments in your proof.

1.108b [2+] Give a combinatorial proof that the number of partitions of $[n]$ such that no two *cyclically consecutive* integers (i.e., two integers i, j for which $j = i + 1 \pmod{n}$) appear in the same block is equal to the number of partitions of $[n]$ with no singleton blocks.

1.175 [2+] For $i, j \geq 0$ and $n \geq 1$, let $f_n(i,j)$ denote the number of pairs (V, W) of subspaces of $\mathbb{F}_q^n$ such that $\dim V = i$, $\dim W = j$, and $V \cap W = 0$. Find a formula for $f_n(i,j)$ which is a power of q times a q-multinomial coefficient.