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Math 245

Homework 5

Due on February 19, 2025

Turn in your **best two problems** from the four problems below.

You may turn in one problem from Homework 4, but at least one problem has to be from this set.

E2 [2+] Let $\Diamond_d$ be the convex hull of $\{\pm\mathbf{e}_i \ : \ i = 1, 2, \ldots, d\}$, where $\mathbf{e}_i$'s are the vectors in the standard basis of $\mathbb{R}^d$. (This polytope is the d-dimensional *cross-polytope*. When $d = 3$ it is an octahedron.)

(1) Find an $\mathcal{H}$-representation of $\Diamond_d$. Justify your answer.

(2) Use a combinatorial argument to find an expression for $i(\Diamond_d, n)$, using the basis $\left\{ \binom{n}{k} \ : \ 0 \le k \le d \right\}$.

(3) Find explicitly the h^*-polynomial of $\Diamond_d$.

E3 [2+] Let $R_d \subset \mathbb{R}^d$ be the convex hull of $\left\{ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d, -\sum_{i=1}^{d} \mathbf{e}_i \right\}$. (This polytope is the d-dimensional *standard reflexive polytope*.)

Find explicitly the h^*-polynomial of R_d.

E4 [2] Let $\mathbf{s} = (s_1, s_2, \ldots, s_d) \in \mathbb{P}^d$ be a sequence of positive integers, and $P_\mathbf{s} \subset \mathbb{R}^d$ the convex hull of $(0, 0, \ldots, 0)$, $(0, 0, \ldots, 0, s_d)$, $(0, \ldots, 0, s_{d-1}, s_d)$, $\ldots$, $(s_1, s_2, \ldots, s_d)$. Show that

$$i(P_\mathbf{s}, n) = i(P_{(s_d, s_{d-1}, \ldots, s_1)}, n).$$

4.52 [3-] Let $P_\mathbf{s}$ be as defined in Problem **E4**. Show that if $\mathbf{s} = (d, d-1, \ldots, 2, 1)$, then

$$i(P_\mathbf{s}, n) = (n+1)^d.$$