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Math 245

Homework 6

Due on February 26, 2025

Turn in your **best two problems** from the six problems below. Parts (a) and (b) of **E9** are treated as two separate problems; however, you can only choose *one* of them to submit since part (a) follows from a solution for part (b).

You may turn in one problem rated as [2+] or [3-] from Homework 5, but at least one problem has to be from this set.

E5 [2+] For each permutation $\pi \in \mathfrak{S}_d$, define the simplex

$$F_\pi := \{\mathbf{x} \in \mathbb{R}^d \ : \ 1 \geq x_{\pi(1)} \geq x_{\pi(2)} \geq \cdots \geq x_{\pi(d)} \geq 0\}, \ \forall \pi \in \mathfrak{S}_d.$$

Let Γ be the triangulation of the unit d-cube induced by the collection $\{F_\pi \ : \ \pi \in \mathfrak{S}_d\}$, meaning that each F_π is a maximal simplex in Γ. Define G_d as the directed graph on $\mathfrak{S}_d$ where $\sigma \to \pi$ is a directed edge if and only if $\sigma = \pi(i, i+1)$ for some $i \in \mathrm{Des}(\pi)$.

In lecture, we claimed (without a proof) that for any ordering of the facets $\{F_\pi\}$ respecting the rule that F_σ appears before F_π whenever $\sigma \to \pi$ is a directed edge in G_d is a shelling order. Prove this statement formally using the definition of shelling in terms of minimal nonfaces.

E6 [2] Suppose the f-vector of a triangulation is $(f_0, f_1, \ldots, f_d)$. We know that the h-vector $(h_0, h_1, \ldots, h_{d+1})$ is defined by

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{k-i} f_{i-1}, \quad 0 \leq k \leq d+1,$$

where we set $f_{-1} = 1$.

Show that it is equivalent to defining the f-vector in terms of h-vector by

$$f_{k-1} = \sum_{i=0}^{k} \binom{d+1-i}{k-i} h_i, \quad 0 \leq k \leq d+1.$$

(This shows that knowing the h-vector is equivalent to knowing the f-vector.)

E7 [2+] Suppose Γ is a shellable triangulation of a d-polytope P with shelling numbers $\{\alpha_i\}$. Prove that the h-vector of Γ is given by

$$h_k = \#(\alpha_i : \alpha_i = k).$$

E8 [3-] Use multivariate generating functions to show

$$\sum_{n\geq 0} (\mathbf{n+1})^d t^n = \frac{\sum_{\pi\in\mathfrak{S}_d} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}}{\prod_{j=0}^d (1-tq^j)}.$$

Here $\mathbf{n+1} = 1 + q + q^2 + \cdots + q^n$ is the q-analogue of $n+1$.

E9 Let

$$P_d = \left\{ (x_1, \dots, x_{2d}) \in \mathbb{R}^{2d} \ : \ \begin{array}{c} 0 \leq x_i \leq 1, \ \forall 1 \leq i \leq 2d, \\ x_1 \geq x_2 \geq \cdots \geq x_d, \\ x_{d+1} \geq x_{d+2} \geq \cdots \geq x_{2d} \end{array} \right\}.$$

(a) [2] Find an explicit formula for the volume of P_d.

(b) [2+] Give a shellable unimodular triangulation for P_d, using which to describe the h^*-vector of P_d.

E10 [2+] Let $P \subset \mathbb{R}^D$ be a d-polytope with h^*-vector $(\delta_0, \delta_1, \dots, \delta_d)$. (So $h^*(P,x) = \sum_{i=0}^d \delta_i x^i$.) Suppose for any $i : 0 \leq i \leq d$, we have $\delta_i = \delta_{d-i}$. Prove that

$$d \cdot \mathrm{nvol}(P) = \sum_{F\,:\,\text{facet of } P} \mathrm{nvol}(F).$$

Here $\mathrm{nvol}(Q)$ is the normalized volume of a polytope Q.

Hint: Consider the generating function for $i(\partial P, n)$ where ∂P denotes the boundary of P.