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Math 245

Homework 7

Due on March 5, 2025

Turn in your **best two problems** from the six problems below.

You may turn in one problem rated as [2+] or [3-] from Homework 6, but at least one problem has to be from this set.

3.14a [2+] A finite poset P is a *series-parallel poset* if it can be built up from a one-element poset using the operations of disjoint union and ordinal sum. There is a unique four-element poset (up to isomorphism) that is not series-parallel, namely, the zigzag poset Z_4 of Exercise 3.66.

Show that a finite poset P is series-parallel if and only if it contains no induced subposet isomorphic to Z_4. Such posets are sometimes called *N-free posets*.

3.34 [2] Find all nonisomorphic posets P such that

$$F(J(P), x) = (1 + x)(1 + x^2)(1 + x + x^2).$$

3.44a [3-] Let $\omega = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. Let $P_\omega = \{(i, a_i) : i \in [n]\}$, regarded as a subposet of $\mathbb{P} \times \mathbb{P}$. In other words, define $(i, a_i) \leq (k, a_k)$ if $i \leq k$ and $a_i \leq a_k$. Let $j(P)$ denote the number of order ideals of the poset P. Show that

$$\sum_{\omega \in \mathfrak{S}_n} j(P_\omega) = \sum_{i=0}^{n} \frac{n!}{i!} \binom{n}{i}.$$

3.45a [2] Let $L_k(n)$ denote the number of k-element order ideals of the boolean algebra B_n. Show that for fixed k, $L_k(n)$ is a polynomial function of n of degree $k - 1$ and leading coefficient $1/(k - 1)!$. Moreover, the differences $\Delta^i L_k(0)$ are all nonnegative integers.

3.127ab [2+]

(a) How many maximal chains does Π_n have?

(b) The symmetric group $\mathfrak{S}_n$ acts on the partition lattice Π_n in an obvious way. This action induces an action on the set $\mathcal{M}$ of maximal chains of Π_n. Show that the number $\#\mathcal{M}/\mathfrak{S}_n$ of $\mathfrak{S}_n$-orbits on $\mathcal{M}$ is equal to the Euler number E_{n-1}. For instance, when n $= 5$ a set of orbit representatives is given by (omitting $\hat{0}$ and $\hat{1}$ from each chain, and writing e.g. 12-34 for the partition whose nonsingleton blocks are $\{1,2\}$ and $\{3,4\}$): $12 < 123 < 1234, 12 < 123 <$ 123-45, $12 <$ 12-34 $<$ 125-34, $12 <$ 12-34 $<$ 12-345, $12 <$ 12-34 < 1234.

Hint: use Proposition 1.6.2.

3.178 [2+] Let L_n denote the lattice of faces of an n-dimensional cube, ordered by inclusion, and let $f(n)$ be the total number of chains containing $\hat{0}$ and $\hat{1}$ in L_n. Show that

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \frac{e^x}{2 - e^{2x}}.$$