Fu Liu
Math 245

# Homework 8

Due on March 12, 2025

Turn in your **best two problems** from the six problems below.

You may turn in one problem rated as [2+] or [3-] from Homework 7, but at least one problem has to be from this set.

**P1** [2+] In class, we discussed that given a vector space or an affine space $V$, and $S$ a finite subset of $V$, the poset

$$L(S) := \{S \cap W : \; W \text{ is a subspace of } V\}, \text{ ordered by inclusion}$$

is a geometric lattice.

We also showed that $\Pi_n$ is a geometric lattice.

Find $V$ and $S$ such that $\Pi_n \cong L(S)$. Justify your answer.

**3.54** [2] Let $P$ be a finite poset. Simplify the sum

$$f(P) = \sum_{t_1 < \cdots < t_n} \frac{1}{(\# \bigvee_{t_1} - 1) \cdots (\# \bigvee_{t_{n-1}} - 1)},$$

where the sum ranges over all nonempty chains of $P$ for which $t_n$ is a maximal element of $P$.

**3.57ab** [2+] Let $P$ be an $n$-element poset. If $t \in P$ then set $\lambda_t = \#\{s \in P : s \leq t\}$. Show that

$$e(P) \geq \frac{n!}{\prod_{t \in P} \lambda_t},$$

where the equality holds if and only if every component of $P$ is a rooted tree (where the root as usual is the maximum element of the tree).

**3.62ab** [2] Let $P_n$ be the poset with elements $s_i$, $t_i$ for $i \in [n]$, and cover relations $s_1 < s_2 < \cdots < s_n$ and $t_i > s_i$ for all $i \in [n]$. (E.g, $P_3$ has the Hasse diagram of Figure 3.46 in the book.)

**(a)** Find a "nice" expression for the rank-generating function $F(J(P_n), x)$.

**(b)** Let $P = \lim_{n \to \infty} P_n$. Find the rank-generating function $F(J_f(P), x)$.

**3.58** [3-] Let $P$ be a finite set. Let $A$ be an antichain of $P$ which intersects every maximal chain. Show that

$$e(P) = \sum_{t \in A} e(P - t).$$

Give an elegant bijective proof.

**3.94** [2+] Let $L = J(P)$ be a finite distributive lattice. A function $v : L \to \mathbb{C}$ is called a *valuation* (over $\mathbb{C}$) if $v(\hat{0}) = 0$ and

$$v(s) + v(t) = v(s \wedge t) + v(s \vee t), \forall s, t \in L.$$

Prove that $v$ is uniquely determined by its values on the join-irreducibles of $L$ (which we may identify with $P$). More precisely, show that if $I$ is an order ideal of $P$, then

$$v(I) = -\sum_{t \in I} v(t)\mu(t, \hat{1}),$$

where $\mu$ denotes the Möbius function of $I$ (considered as a subposet of $P$) with $\hat{1}$ adjoined.

**3.144** [2] Let $B_k$ denote a boolean algebra of rank $k$, and $\Omega(B_k, m)$ its order polynomial. Show that $\Omega(B_{n+1}, 2) = \Omega(B_n, 3)$.