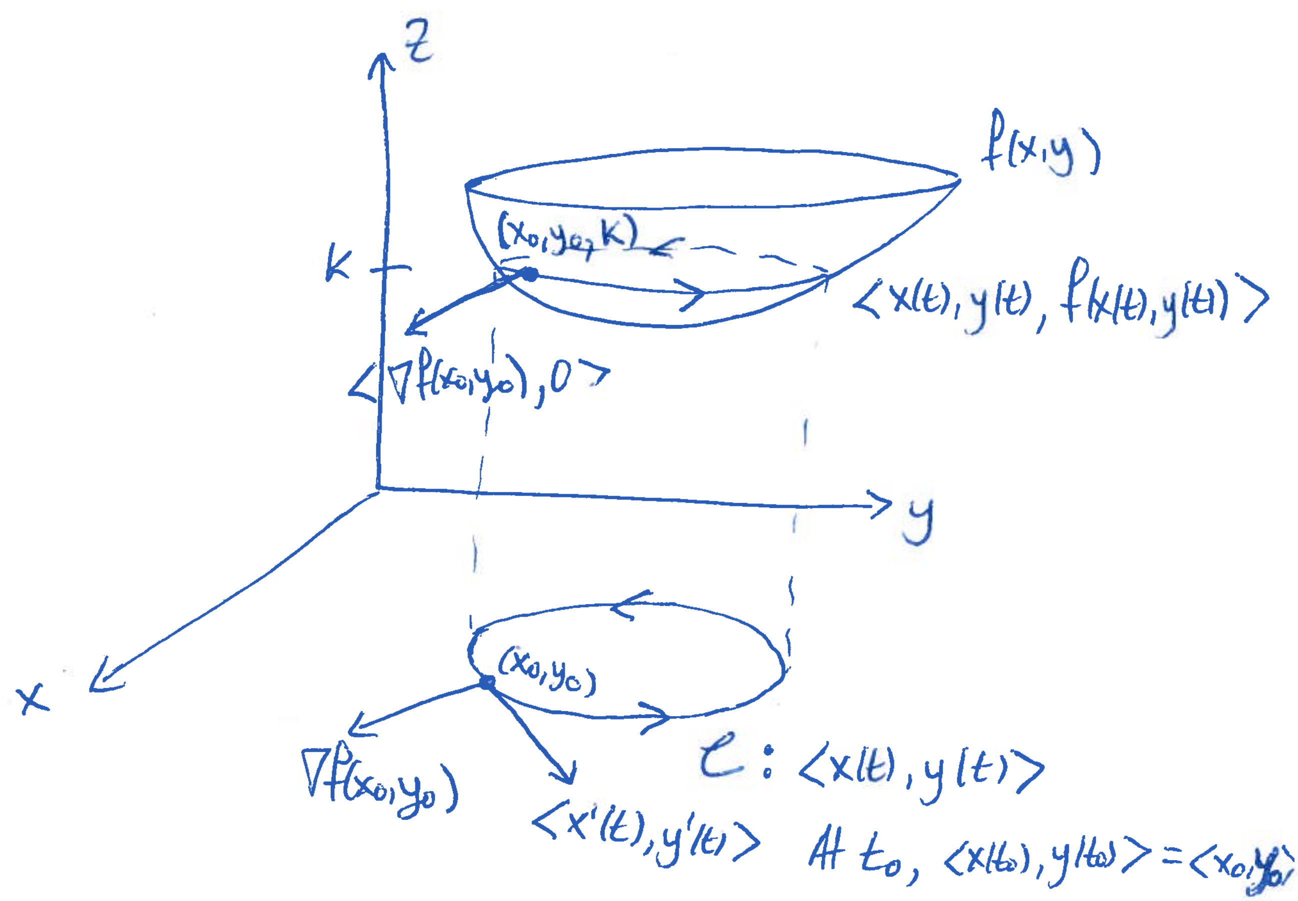
# Directional Derivatives and the Gradient Vector

Gradient vector : $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

Directional derivative : $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$ (spits out a number)

What are these?




How does your altitude change as you travel around $C$?

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \nabla f(x,y) \cdot \langle x'(t), y'(t) \rangle$$

How is your altitude change as you travel around the surface $z=f(x,y)$?

Stays the same! $\Rightarrow \frac{dz}{dt}=0$ !

$$\Rightarrow \quad 0 = \nabla f(x,y) \cdot \langle x'(t), y'(t) \rangle$$

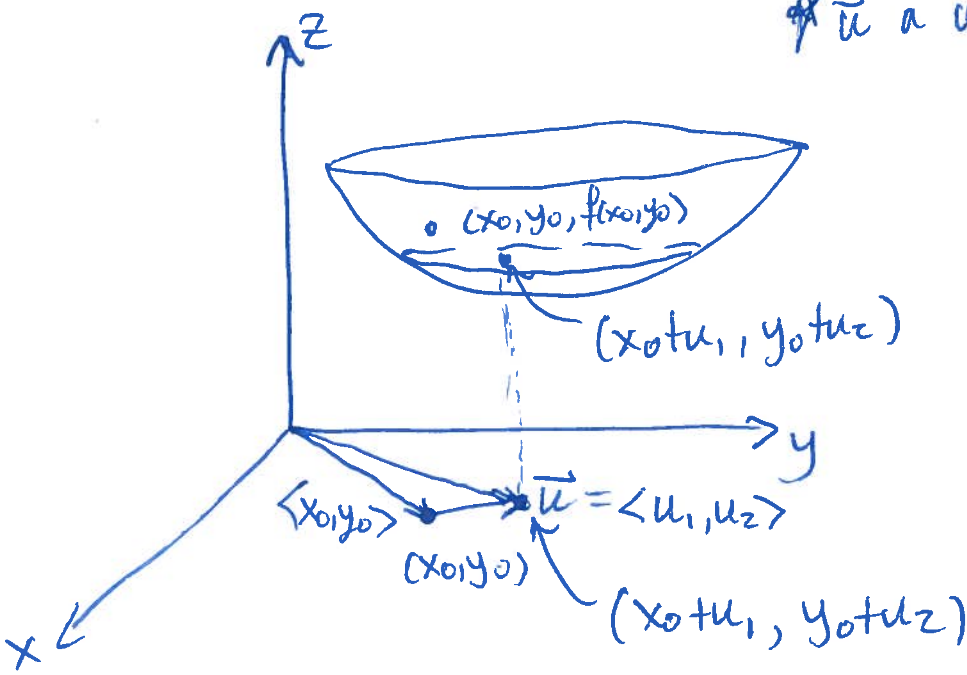
So $\nabla f(x,y)$ is normal to $C$.
(perpendicular to your velocity)

☆ Gradient points in direction of steepest ascent

Directional derivative: $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$

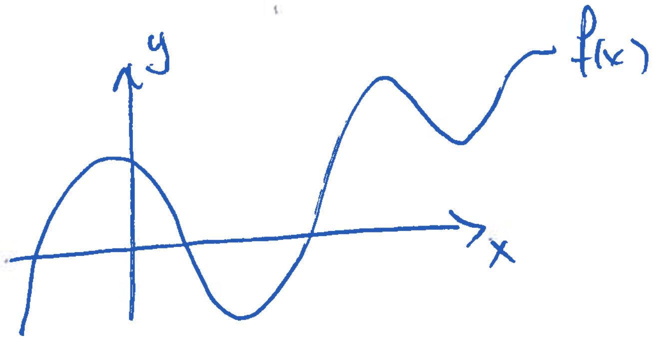
"How does f change if you travel in the $\vec{u}$ direction"

☆ $\vec{u}$ a unit vector



# Optimization

## 17A




Find local max/min: 1. Find critical points

2. Determine which are max/min

Find absolute max/min: 3. Pick the larges/smallest value of $f$.

If on an finite interval, need to also check endpoints

## Now back to 17C,

Same idea, but slightly more complicated:

1. 2nd derivative test ( $f_x = 0$ and $f_y = 0$ )

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

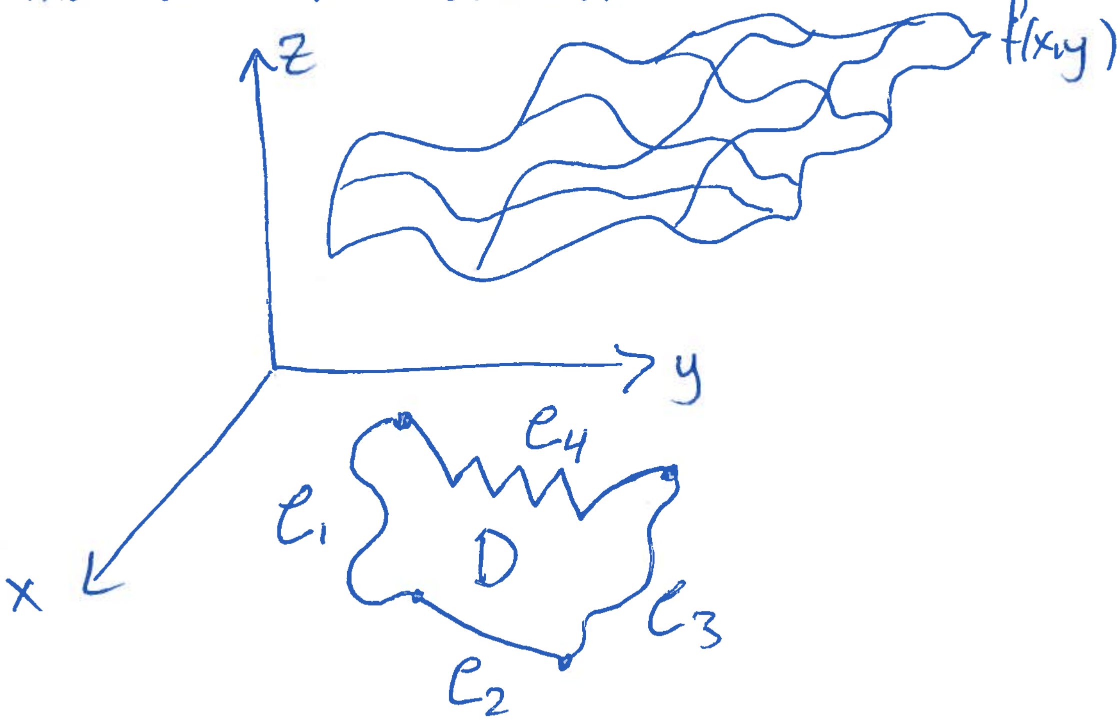
(a) $D > 0$ and $f_{xx} > 0 \longrightarrow$ local min

(b) $D > 0$ and $f_{yy} < 0 \longrightarrow$ local max

(c) $D<0 \rightarrow$ saddle

(d) $D=0 \rightarrow$ inconclusive

2. Endpoints are now boundaries




After find the critical points of $f$ in $D$, we need to investigate $f$ on the boundary.

Say on $e_1$, ~~xxx~~ we know the points are given by $(x,y)$ where $y=g(x)$. Then, along $e_1$, $f$ is $f(x,y)=f(x,g(x))$.

$\rightarrow$ Then use 17A techniques.

# Method of Lagrange Multipliers

maximize $f(x)$

Subject to $g_1(x)=0,\ g_2(x)=0, \cdots, g_m(x)=0$

$$\nabla f(x) = \sum_{k=1}^{m} \lambda_k \nabla g_k(x) \iff \nabla f(x) - \sum_{k=1}^{M} \lambda_k \nabla g_k(x) = 0$$

$$\mathcal{L} = f - \sum_{k=1}^{m} \lambda_k g_k$$

$$\nabla \mathcal{L} = 0 \iff \begin{cases} \nabla f - \sum \lambda_k g_k = 0 \\ g_1(x) = g_2(x) = \cdots = g_m(x) = 0 \end{cases}$$