Systems of Nonlinear Differential Equations

Autonomous nonlinear DEs

$$\vec{x}\,' = \vec{F}(\vec{x})$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \vec{x}\,' = \begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix}$$

$$\vec{F}(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

Autonomous linear DE

$$\vec{x}\,' = A\vec{x} + \vec{g}$$

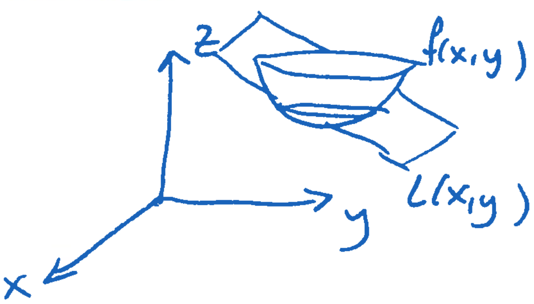
Equilibria: $f_1(x_1, x_2) = 0$ <u>and</u> $f_2(x_1, x_2) = 0$

Local Stability Analyses

We will use the linearization of $\vec{F}$ at an equilibria, $\vec{x}^*$, to determine the local behavior.

Recall

Linearization of $f(x,y)$ at (x_0, y_0)



$$L(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0) \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Now, we have a vector-valued function, so there are "two functions to linearize"

$$\vec{F}(\vec{x}) \approx \vec{F}(\vec{x}_0) + \overrightarrow{DF}(\vec{x}_0) \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$$= \vec{F}(\vec{x}_0) + \begin{bmatrix} \frac{\partial f_1}{\partial x}(x_0, y_0) & \frac{\partial f_1}{\partial y}(x_0, y_0) \\ \frac{\partial f_2}{\partial x}(x_0, y_0) & \frac{\partial f_2}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Now, if we take $\vec{x}_0 = \vec{x}^*$ (an equilibria), then

$$\vec{F}(\vec{x}_0) = \vec{F}(\vec{x}_0^*) = \vec{0} \quad !!!$$

So, near the equilibria $\vec{x}^*$,

$$\vec{F}(\vec{x}) \approx \underbrace{\overrightarrow{DF}(\vec{x}^*)}_{\text{Jacobian matrix}} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix} .$$

Then, near the equilibria

$$\vec{x}' \approx \overrightarrow{DF}(\vec{x}^*) \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}$$

So we have a linear system whose behavior near the equilibrium can be determined by the eigenvalues of $\overrightarrow{DF}(\vec{x}^*)$.

<u>Note</u>: This does not work if you have a center!!

<u>Ex</u> Lotka-Volterra predator-prey

$$\frac{dR}{dt} = rR - aRW$$

$$\frac{dW}{dt} = -KW + bRW$$

R - prey population

W - predator population

r - per capita reproductive rate of prey

K - per capita death rate of predator

a - ~~average~~ rate of consumption of prey by predators

b - rate at which consumption is converted into predator offspring.