Math 127A, Fall 2019.

HW 2 Solutions (mostly adapted from Abbott's Instructor's Manual)

1.3.3 (a). The point of this problem is that existence of the greatest lower bound follows form the Axiom of Completeness, that is, from the existence of the least upper bound.

Because A is bounded below, $B \neq \emptyset$. The first observation is that, for all $a \in A$ and $b \in B$, we have $b \leq a$. This tells us is that B is bounded above and thus $\beta = \sup B$ exists by the Axiom of Completeness.

Fix an $a \in A$. We have observed that a is an upper bound for B; we conclude that $\beta = \sup B \leq a$. As this is true for every $a \in A$, β is a lower bound for A. Thus $\beta \in B$, and so $\beta = \max B$. This last fact states exactly that β is the largest of all lower bounds for A, i.e., that $\beta = \inf A$. In particular, $\inf A$ exists.

Remark. Another way to solve this problem is to show first that $\inf A$ exists. One way to do this is by considering the set $-A = \{-a : a \in A\}$. Then we need a few verifications (which are good exercises). First we check that, as A is bounded below, $-A$ is bounded above. Thus $\sup(-A)$ exists. Then we check that $-\sup(-A)$ is the greatest lower bound for A, that is, that $\inf A$ exists and $\inf A = -\sup(-A)$. Once we know that $\inf A$ exists, Problem 1.3.3 (a) becomes easy: by definition of greatest lower bound, $\inf A = \max B$, and then since $\max B$ exists it must equal to $\sup B$.

1.3.7. Since a is an upper bound for A, we only need to verify the second part of the definition of supremum and show that if d is any upper bound for A then $a \leq d$. However, by the definition of upper bound, $x \leq d$ for every $x \in A$, in particular for $x = a$.

1.3.8. *Note*. We assume $\mathbb{N} = \{1, 2, \ldots\}$.
(a) The supremum is 1 and the infimum is 0.
(b) The supremum is 1 and the infimum is -1.
(c) The supremum is $1/3$ and the infimum is $1/4$.
(d) The supremum is 1 and the infimum is 0.

1.3.9. (a) Let $\epsilon = \sup B - \sup A > 0$. Then there exists an element $b \in B$ satisfying $\sup B - \epsilon < b$, which implies $\sup A < b$. Further, for every $a \in A$, $a \leq \sup A \leq b$ so b is an upper bound for A.
(b) Let $A = B = (-\infty, 1)$. Then $\sup A = \sup B = 1$, but no element of $B = A$ is an upper bound for A, because A does not have a maximum (as $\sup A \notin A$).

1.3.11. (a) True. We need to show that $\sup B$ is an upper bound for A, which it is because it is an upper bound for B.
(b) True. Let $c \in \mathbb{R}$ satisfy $\sup A < c < inf B$. Then, for any $a \in A$, $a \leq \sup A < c$. Also, for any $b \in B$, $c < \inf B \leq B$.
(c) False. Consider $A = (0, 1)$ and $B = (1, 2)$. Then $c = 1$ is such a number, $\sup A = 1 = \inf B$.

1.4.1. (a) omitted. For (b), argue by contradiction: if $a + t$ is rational, then $t = (a + t) - a$ is also rational (and similarly for the product). For (c), the answer is no for both addition and multiplication. Consider $a = \sqrt{2}$, $b = -a$. Then a and b are irrational, but $a + b = 0$ and $ab = -2$ are both rational.

1.4.2. To show that s is an upper bound for A, assume it is not, i.e., that there exists an $a \in A$ with

$s < a$. Then, by the Archimedian property, there exists an $n \in \mathbb{N}$ so that $1/n < a - s$, thus $s + 1/n < a$ and $s + 1/n$ is not an upper bound, which contradicts the assumption. To show that s is the least upper bound, again assume it is not, that is, that $t < s$ is an upper bound. Then there exists an $n \in \mathbb{N}$ such that $t + 1/n < s$, so $s - 1/n > t$, and $s - 1/n$ is also an upper bound, which again contradicts the assumption.

1.4.3. Denote $A = \cap_{n=1}^{\infty}(0, 1/n)$. Take an arbitrary $x \in \mathbb{R}$. We need to show that $x \notin A$. This is clear if $x \leq 0$ as then $x \notin (0, 1/n)$ for every $n \in \mathbb{N}$. If $x > 0$, then we know there exists an $n_0 \in \mathbb{N}$ such that $1/n_0 < x$, and so $x \notin (0, 1/n_0)$. This implies $x \notin A$ and our proof is complete.

1.4.5. We have to show the existence of an irrational number between any two real numbers a and b with $a < b$. Apply Theorem 1.4.3 for the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$ to find a rational number r satisfying $a - \sqrt{2} < r < b - \sqrt{2}$. Then $a < r + \sqrt{2} < b$ and $r + \sqrt{2}$ is irrational by Exercise 1.4.2(b).