Math 135A, Fall 2025.

Homework 6

1. Assume that weekly sales of diesel fuel at a gas station are X tons, where X is a random variable with density function

$$f(x) = \begin{cases} c(1-x)^4 & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute c. (b) Compute EX. (c) The current tank (of capacity 1) will be emptied and out for repairs next week; to save money, the station wants to rent a tank for the week with capacity just large enough that its supply will be exhausted with probability 0.01. What is the capacity of the tank the station needs to rent?

2. Alice arrives at a train station at a time uniformly distributed between 8 p.m. and 9 p.m., and boards the first train that arrives. There are two types of trains that arrive between 7 a.m. and midnight: those that head to town A arrive every 15 minutes starting at 7 a.m., and those that head to town B arrive every 15 minutes starting at 7:05 a.m.. (a) How long does Alice wait, on the average? (b) What proportion of days does she go to A?

3. If X is an Exponential(1) random variable, compute the density of (a) $Y = \log X$ and (b) $Z = (\log X)^2$.

4. Assume A is uniform in $[0, 5]$. Compute the probability that the equation $4x^2 + 4Ax + A + 2 = 0$ has two real roots.

5. January snowfall, in inches, in Truckee, California has expectation 50 and standard deviation 35. (These are close to true numbers, from the ranger station website.) Assume normal distribution and year to year independence. What is the probability that, starting from the next January, it will take at least 11 years to get January snowfall over 100 inches?

6. A fair die is rolled 1000 times. Let A be the event that the number of 6's is in the interval $[150, 200]$, and B the event that the number of 5's is exactly 200. (a) Approximate $P(A)$. (b) Approximate $P(A \mid B)$.

You should also do the five Problems in Section 6 of the book.

Solutions

1. Solve $\int_0^1 f(x)\,dx = 1$ to get $c = 5$. (b) Compute $\int_0^1 xf(x)\,dx = \frac{1}{6}$. (c) Solve $P(X \geq a) = \int_a^1 f(x)\,dx = 0.01$, $(1-a)^5 = 0.01$, $a = 1 - (0.01)^{1/5} \approx 0.6019$.

2. (a) Assume X is uniform on $[0, 15]$. The waiting time W is $5 - X$ if $X \leq 5$ and $15 - X$ otherwise. The answer is $\frac{1}{15}(\int_0^5 (5-x)\,dx + \int_5^{15}(15-x)\,dx) = 25/6$ minutes. (b) If she arrives between 8 and 8:05 she goes to B, between 8:05 and 8:15 she goes to A, etc., so the answer is 2/3.

3. (a) $f_Y(y) = e^{-e^y} \cdot e^y$, $y \in \mathbb{R}$. For (b), assume $z > 0$ and compute

$$P(Z \leq z) = P((\log X)^2 \leq z) = P(-\sqrt{z} \leq \log X \leq \sqrt{z}) = P(e^{-\sqrt{z}} \leq X \leq e^{\sqrt{z}}) = e^{-e^{-\sqrt{z}}} - e^{-e^{\sqrt{z}}},$$

and then differentiate to get

$$f_Z(z) = \frac{1}{2\sqrt{z}}\left(e^{-e^{-\sqrt{z}}} \cdot e^{-\sqrt{z}} + e^{-e^{\sqrt{z}}} \cdot e^{\sqrt{z}}\right).$$

4. The event happens if and only if the discriminant of the quadratic equation is nonnegative, so we need to compute

$$P((4A)^2 - 4 \cdot 4(A+2) > 0) = P(A^2 - A - 2 > 0) = P((A+1)(A-2) > 0) = P(A > 2) = \frac{3}{5}.$$

5. Let X be the snowfall in a year. Then

$$P(X \leq 100) = P\left(\frac{X-50}{35} \leq \frac{100-50}{35}\right) = P\left(Z \leq \frac{10}{7}\right) = \Phi\left(\frac{10}{7}\right) \approx 0.9234,$$

and so the answer is $\Phi(\frac{10}{7})^{10}$ or approximately 0.4509.

6. For (a), let S be Binomial$(1000, 1/6)$, with $\mu = 1000/6$ and $\sigma = \sqrt{5000/36}$. Then

$$P(150 \leq S \leq 200) = P\left(\frac{150-\mu}{\sigma} \leq \frac{S-\mu}{\sigma} \leq \frac{200-\mu}{\sigma}\right) \approx \Phi\left(\frac{200-\mu}{\sigma}\right) - \Phi\left(\frac{150-\mu}{\sigma}\right) \approx 0.9190.$$

For (b), repeat with S Binomial(800, 1/5), thus $\mu = 800/5$ and $\sigma = \sqrt{3200/25}$, to get 0.8114.