Math 135A, Fall 2025.

Homework 7

1. A fair die is rolled five times. (a) Compute the joint p.m.f. of X and Y, where X is the first number rolled and Y is the largest of the five numbers rolled. (b) Compute the joint p.m.f. of X and Y, where X is the number of 1's rolled and Y is the largest of the five numbers rolled. (c) Compute the joint p.m.f. of X and Y, where X is the first number rolled and Y is the number of 1's rolled.

2. Joint density of (X, Y) is given by

$$f(x,y) = c(x^2 + \frac{xy}{2}), \quad 0 < x < 1,\, 0 < y < 2.$$

(a) Compute c. (b) Compute the density of X. (c) Compute $P(X > Y)$. (d) Compute the conditional probability $P(Y > 1/2 | X < 1/2)$. (e) Find EY.

3. Mr. Smith arrives at a location at a time uniformly distributed between 12:15 and 12:45, while Mrs. Smith independently arrives at the same location at a time uniformly distributed between 12 and 1 (all times p.m.). (a) Compute the probability that the first person to arrive waits no longer than 5 minutes. (b) Compute the probability that Mr. Smith arrives first.

4. Joint density of (X, Y) is given by

$$f_1(x,y) = \begin{cases} x \cdot e^{-(x+y)} & x, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent? (b) Repeat with joint density

$$f_2(x,y) = \begin{cases} 2 & 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

You should also do the five Problems in Section 7 of the book, other than 5(a).

Solutions

1. (a) For $i, j = 1, \dots, 6$, the event $\{X = i, Y = j\}$ happens: if $i = j$, when the first roll is i and the remaining 4 rolls are in $[1, i]$; and if $i < j$, when the first roll is i, the remaining 4 rolls are in $[1, j]$ but not all 4 are in $[1, j-1]$. Therefore,

$$P(X = i, Y = j) = \begin{cases} \frac{1}{6} \cdot \left(\frac{i}{6}\right)^4 & \text{if } i = j, \\ \frac{1}{6} \cdot \frac{j^4 - (j-1)^4}{6^4} & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases}$$

(b) For $i = 0, \dots, 5$, $j = 1, \dots, 6$, let's look at when the event $\{X = i, Y = j\}$ happens. If $j = 1$, then it only happens when $i = 5$, that is, when all rolls are 1. If $j \geq 2$, then $i < 5$ (not all rolls can be 1) and all of the $5 - i$ rolls that are not 1 are in $[2, j]$, but not all of these $5 - i$ rolls are in $[2, j-1]$. Therefore, For $i = 0, \dots, 5$, $j = 1, \dots, 6$,

$$P(X = i, Y = j) = \begin{cases} \frac{1}{6^5} & \text{if } i = 5, j = 1, \\ \frac{1}{6^5} \cdot \binom{5}{i} \left((j-1)^{5-i} - (j-2)^{5-i}\right) & \text{if } i < 5, j \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(c) For $i = 1, \dots, 6$, $j = 0, \dots 5$, the event $\{X = i, Y = j\}$ happens: if $i = 1$ and $j \geq 1$, when the first roll is i and there is $j - 1$ 1s on the remaining 4 rolls are in $[1, i]$; and if $i > 1$ and $j \leq 4$, when the first roll is i, and there is j 1s on the remaining 4 rolls. Therefore,

$$P(X = i, Y = j) = \begin{cases} \frac{1}{6^5} \cdot \binom{4}{j-1} \cdot 5^{5-j} & \text{if } i = 1, j \geq 1, \\ \frac{1}{6^5} \cdot \binom{4}{j} \cdot 5^{4-j} & \text{if } i > 1, j \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

2. All detailed computations omitted, but you should do them! (a) Solve $\int_0^1 dx \int_0^2 f(x, y)\, dy = 1$ to get $c = \frac{6}{7}$. (b) $f_X(x) = \int_0^2 f(x, y)\, dy$. (c) $\int_0^1 dx \int_0^x f(x, y)\, dy$.
(d)

$$\frac{\int_0^{1/2} dx \int_{1/2}^2 f(x, y)\, dy}{\int_0^{1/2} dx \int_0^2 f(x, y)\, dy}.$$

(e) $\int_0^1 dx \int_0^2 y f(x, y)\, dy$.

3. Let the unit be 1 hour, X uniform on $[1/4, 3/4]$, Y uniform on $[0, 1]$ and independent. (a) $P(|X - Y| < 1/12) = 1/6$ (draw a picture). (b) $P(X \leq Y) = 1/2$.

4. (a) Yes, with marginal densities $f_X(x) = x \cdot e^{-x}$ and $f_Y(y) = e^{-y}$. (b) No, (X, Y) is distributed uniformly, but not on a rectangle.