Math 135A, Fall 2025.

Homework 8

1. Joint density of (X, Y) is given by

$$f(x, y) = xe^{-x(y+1)}, \quad x, y > 0.$$

Compute the density of $Z = XY$.

2. Assume that $X_1, \ldots, X_{10}$ are independent Exponential(λ) random variables. Compute the densities of $H = \max\{X_1, \ldots, X_{10}\}$ and $L = \min\{X_1, \ldots, X_{10}\}$.

3. Select a point (X, Y) at random from the square $[-1, 1] \times [-1, 1]$. Compute (a) $E(|X| + |Y|)$, (b) $E|XY|$, and (c) $E|X - Y|$.

4. Suppose that Alice and and Bob each randomly, and independently, choose 3 out of 10 objects without replacement. Find the expexted number of objects that are (a) chosen by both Alice and Bob, (b) not chosen by either Alice or Bob, (c) chosen by exactly one of them.

5. A coin has probability p of Heads. Toss this coin n times, and let X be the number of tosses, from toss 2 on, that have different outcome than the previous toss. Compute EX.

6. Let (X, Y) be a random point of in the square $\{(x, y) : 0 \leq x, y \leq 1\}$. (a) Compute the density of $Z = XY$, EZ, and $\mathrm{Var}(Z)$. (b) Assume that 1000 such points (X_i, Y_i), $i = 1, \ldots, 1000$, are chosen independently, and approximate the probability $P(X_1Y_1 + X_2Y_2 + \cdots + X_{1000}Y_{1000} < 255)$.

You should also do the six Problems in Section 8 of the book.

Solutions

1. For $z \geq 0$, $P(Z \leq z) = P(Y \leq z/X) = \int_0^\infty dx \int_0^{z/x} f(x,y)\,dy = 1 - e^{-z}$, so that $f_Z(z) = e^{-z}$, Exponential(1).

2. $P(H \leq h) = P(X_1 \leq h)^{10} = (1 - e^{-\lambda h})^{10}$ and so

$$f_H(h) = 10\lambda \cdot e^{-\lambda h}(1 - e^{-\lambda h})^9.$$

Moreover, $P(L \leq \ell) = 1 - P(L \geq \ell) = 1 - e^{-10\lambda\ell}$, and so

$$f_L(\ell) = 10\lambda \cdot e^{-10\lambda\ell},$$

so L is Exponential(10λ).

3. (a) $|X|$ is Uniform on $[0,1]$, so $E|X| = \frac{1}{2}$, and the answer is 1. (b) By independence, the answer is $\frac{1}{4}$. (c)$\frac{1}{2}\int_{-1}^1 dx \int_{-1}^x (x-y)\,dy = \frac{2}{3}$.

4. In all three parts, we use the indicator trick. For (a), let X be the number of objects that are chosen by both Alice and Bob. Write $I = I_1 + \ldots + I_{10}$, where $I_i = I_{\{\text{object } i \text{ is chosen by both Alice and Bob}\}}$. By additivity and symmetry,

$$EX = 10 \cdot EI_1 = 10 \cdot P(\text{object 1 is chosen by both Alice and Bob}) = 10 \cdot (3/10)^2,$$

and so $EX = 10 \cdot (0.3)^2 = 0.9$. By very similar reasoning, the answer to (b) is $10 \cdot (0.7)^2 = 4.9$, and for (c) $10 \cdot 2 \cdot 0.3 \cdot 0.7 = 4.2$.

5. or $i \geq 2$, let $I_i = I_{\{\text{toss } i \text{ is different from toss } i-1\}}$. Then $I = I_2 + \ldots + I_n$, and

$$EI_i = P(\text{toss } i \text{ is different from toss } i-1) = 2p(1-p)$$

for all $i \geq 2$, and so

$$EX = (n-1) \cdot 2p(1-p).$$

6. (a) For $z \in (0,1)$,

$$P(Z \leq z) = P(XY \leq z) = 1 - \int_z^1 \left(1 - \frac{z}{x}\right)\,dx = z - z\ln z$$

and so $f_Z(z) = -\ln z$. By independence $E(XY) = EX \cdot EY = 1/4$ and $E((XY)^2) = EX^2 \cdot EY^2 = 1/9$, so $EZ = 1/4$ and $\text{Var}(Z) = 7/144$.

(b) Let S be the sum. Then $ES = 250$ and $\text{Var}(S) = 875/18$, so

$$P(S \leq 255) = P\left(\frac{S - 250}{\sqrt{875/18}} \leq \frac{5}{\sqrt{875/18}}\right) \approx \Phi\left(\frac{3}{\sqrt{17.5}}\right) \approx 0.7634.$$