Math 135A, Fall 2025.
Dec. 9, 2025.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

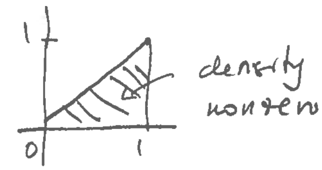
NAME(sign): ______________________________

ID#: ______________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Electronic devices, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions: proper interpretation of exam questions is a part of the exam. You do not need to evaluate $\Phi(z)$, for *positive* z.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
5	
6	
TOTAL	



1. A pair (X, Y) of random variables has joint density

$$f(x,y) = \begin{cases} cxy & \text{if } x \in [0,1],\ y \in [0,1], \text{ and } y \leq x \\ 0 & \text{otherwise} \end{cases}$$

In your answers to questions below, give all numbers as simple reduced fractions. You may use: $\int_0^1 x^n\,dx = \frac{1}{n+1}$ for $n > -1$.

(a) Compute the constant c.

$$c \int_0^1 dx \int_0^x xy\,dy = c\int_0^1 x \int_0^x y\,dy = c\int_0^1 \frac{x}{2}^2\,dx = c \cdot \frac{1}{8}$$

$$\underline{\underline{c = 8}}$$

(b) Determine the marginal density of X.

$$f_X(x) = 8\int_0^x xy\,dy = 8x \cdot \frac{x^2}{2} = 4x^3$$

if $x \in [0,1]$
(and 0 otherwise)

(c) Compute $E\left(\frac{X}{Y}\right)$.

$$E\left(\frac{X}{Y}\right) = 8\int_0^1 dx \int_0^x \frac{x}{\cancel{y}}\,x\cancel{y}\,dy$$

$$= 8\int_0^1 x^3\,dx = \underline{\underline{2}}$$

2. Assume that a bag initially contains 6 balls: 2 red, 2 green and 2 blue balls. At each step, you choose a ball from the bag at random, note its color, but do not put it back into the bag. Instead, you add to the bag two balls, which are of two different colors, and different in color from the color of the removed ball. (For example, if you choose a red ball in the first step, then after the first step the bag will contain one red, three green and three blue balls.)

(a) Compute the probability that in the first three steps, the selected balls are, in order, red, green and blue.

$$\frac{2}{6} \cdot \frac{3}{7} \cdot \frac{4}{8} = \frac{1}{14}$$

2r 2g 2b   1r 3g 3b   2r 2g 4b

(b) Compute the probability that the balls selected on 2nd and 3rd step are, in order, green and blue.

$$P(*gb) = \underbrace{\frac{2}{6}}_{\text{1st red}} \cdot \frac{3}{7} \cdot \frac{4}{8} + \underbrace{\frac{2}{6}}_{\text{1st green}} \cdot \frac{1}{7} \cdot \frac{4}{8} + \underbrace{\frac{2}{6}}_{\text{1st blue}} \cdot \frac{3}{7} \cdot \frac{2}{8}$$

3r 1g 3b   4r 0 4b   3r 3g 1b   4r 2g 2b

(c) Compute the conditional probability that the ball selected in the first step is red, given that the balls selected on 2nd and 3rd step are, in order, green, and blue.

$$\frac{P(rgb)}{P(*gb)} = \frac{2 \cdot 3 \cdot 4}{2 \cdot 3 \cdot 4 + 2 \cdot 1 \cdot 4 + 2 \cdot 3 \cdot 2} = \frac{6}{11}$$

(d) Compute the probability that the two balls selected on 2nd and 3rd step are of different colors.

By symmetry, every ordered selection of colors has the same prob., so the answer is

$$3 \cdot 2 \cdot (\text{prob. in (b)})$$

3. A deck of 28 cards has 4 cards with number i on them, for $i = 1, \ldots, 7$; that is, 4 cards with number 1 on them, 4 cards with number 2 on them, $\ldots$, 4 cards with number 7 on them. Select 6 cards at random (without replacement) from the deck.

For parts (a)–(c), call a number $i \in \{1, 2, 3, 4, 5, 6, 7\}$ a *loner* if i is represented exactly once in your selection. (For example, if your selection consists of numbers 1 1 2 4 4 6, then 2 and 6 are the only loners.)

(a) Compute the probability that the number 1 is a loner.

$$\frac{4 \cdot \binom{24}{5}}{\binom{28}{6}}$$

(b) Compute the probability that the three numbers 1, 2, 3 are all loners.

$$\frac{4^3 \binom{16}{3}}{\binom{28}{6}}$$

(c) Compute the probability that at least one of numbers 1, 2, 3 is a loner.

$A_1 = \{1 \text{ a loner}\}$

$A_2 = \{2 \text{ —"— }\}$

$A_3 = \{3 \text{ —"— }\}$

$P(A_1 \cup A_2 \cup A_3) =$

$$3 \cdot P(A_1) - 3P(A_1 \cap A_2) + P(A_1 \cap A_2 \cap A_3)$$

$$= 3 \cdot \frac{4\binom{24}{5}}{\binom{28}{6}} - 3 \cdot \frac{4^2\binom{20}{4}}{\binom{28}{6}} + \frac{4^3\binom{16}{3}}{\binom{28}{6}}$$

Problem 3, continued

(d) (Unrelated to (a)–(c).) Let X be the number of 1s in your selection and Y the number of 2s in your selection. Write down the joint probability mass function of (X, Y). (Write a formula rather than table.)

$$P(X=i, Y=j) = \frac{\binom{4}{i}\binom{4}{j}\binom{20}{6-i-j}}{\binom{28}{6}}$$

$i = 1, \ldots, 4$

$j = 1, \ldots, 4$

$i+j \leq 6$

4. Each year, a campus society of 15 students is formed, composed of three groups: 4 Architecture students, 5 Botany students, and 6 Composition students. Each year (independently from year to year) a committee of 3 students is selected at random from among the 15 students. In any particular year, we say that *Architecture wins* if all 3 members of the committee are from Architecture, *Botany wins* if the three are from Botany, and *Composition wins* if the three are from Composition.

(a) Compute the probability that Botany wins next year.

$$\frac{\binom{5}{3}}{\binom{15}{3}}$$

(b) Compute the probability that one of the three groups wins next year.

$$\frac{\binom{4}{3}+\binom{5}{3}+\binom{6}{3}}{\binom{15}{3}}$$

(c) Let N be the number of years, starting from next year, up to (and including) the first year in which one of the three groups wins for the first time. Identify the distribution of N and compute EN.

N is Geometric (p), where p is the prob. from (b)

So $EN = \dfrac{\binom{15}{3}}{\binom{4}{3}+\binom{5}{3}+\binom{6}{3}}$.

(d) Compute the probability that the first group to ever win is Botany.

$P($ B wins 1st year $\mid$ some group wins 1st year$)$

$$= \frac{\binom{5}{3}}{\binom{4}{3}+\binom{5}{3}+\binom{6}{3}}$$

(e) (Unrelated to (a)–(d).) Compute the expected number of Composition students on the committee on any particular year. (As a simple fraction — no sum.)

Label comp. student $1, \dots, 6$, $I_i = I_{\{i\text{th student on committee}\}}$

$X =$ no. of C-students on committee

$EX = E(I_1 + \cdots + I_6) = 6\, P(\text{student 1 on comm.})$

$= 6 \cdot \frac{3}{15} = \underline{\underline{\frac{6}{5}}}$.

5. A deck of 10 cards, consisting of 3 hearts cards (♡) cards, 3 diamonds (♢) cards, and 4 spades (♠) cards, is shuffled until the cards are in random order. Bob then bets that the three suits are together in the deck, that is, the 3 hearts cards are together (next to each other) in the deck, the 3 diamonds cards are together in the deck, and the 4 spades cards are together in the deck.

(a) Compute the probability that Bob wins his bet. Write the answer as a fraction with numerator 1.

$$P(B \text{ wins}) = \frac{3!\,3!\,3!\,4!}{10!} = \frac{\cancel{6}\cdot\cancel{6}\cdot 6\cdot 2\cdot 3\cdot\cancel{4}}{10\cdot 9\cdot 8\cdot 7\cdot\cancel{6}\cdot 5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}$$

$$= \frac{1}{10\cdot 2\cdot 7\cdot 5} = \frac{1}{700}$$

(b) Assume that Bob plays this game once a day for 350 days. Let X be the number of days on which he wins his bet. Identify the distribution of X (as one of the famous probability mass functions.)

X is Binomial $\left(350, \frac{1}{700}\right)$

(b) Assume that X is as in (b). Using an appropriate approximation, estimate the probability that there are at least 3 such days, that is, estimate $P(X \geq 3)$.

$$X \approx \text{Poisson}\left(350\cdot\frac{1}{700}\right) = \text{Poisson}\left(\frac{1}{2}\right)$$

$$P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$$

$$= 1 - e^{-1/2} - \frac{1}{2}e^{-1/2} - \frac{\left(\frac{1}{2}\right)^2}{2!}e^{-1/2}$$

$$= 1 - \frac{13}{8}e^{-1/2}$$

6. Alice and Bob play the following game every day. Alice tosses 5 fair coins. (All coin tosses are independent, as usual.) She wins $-\$1$ (that is, she pays \$1 to Bob) if she tosses 2 or fewer Heads, wins \$1 from Bob if she tosses 3 Heads, wins \$2 if she tosses 4 Heads, and wins \$4 if she tosses 5 Heads. Started from the first day, let Alice's dollar winnings each day be $X_1, X_2, X_3, \ldots$ and let $S_n = X_1 + \cdots + X_n$ be the combined dollar amount of winnings after n days.

(a) Determine the probability mass function of X_1, and compute EX_1 and $\mathrm{Var}(X_1)$. Write the results as simple fractions.

i	$P(X_1 = i)$
-1	$16/2^5$
1	$10/2^5 = \frac{\binom{5}{3}}{2^5}$
2	$5/2^5 = \frac{\binom{5}{4}}{2^5}$
4	$1/2^5$

$$EX_1 = \frac{-16+10+10+4}{2^5} = \frac{8}{2^5} = \frac{1}{4}$$

$$EX_1^2 = \frac{16+10+20+16}{2^5} = \frac{62}{2^5} = \frac{31}{16}$$

$$\mathrm{Var}(X_1) = \frac{31}{16} - \frac{1}{16} = \frac{30}{16} = \frac{15}{8}$$

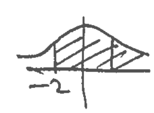
(b) Using a relevant approximation, estimate the probability that Alice's combined winnings after $n = 120 = 4 \cdot 30$ games are nonnegative, that is, that $S_n \geq 0$. (Your answer may involve Φ evaluated at a positive number, which you do not need to evaluate.)

$$P(S_n \geq 0) = P\left(\frac{S_n - \frac{1}{4}n}{\sqrt{\frac{15}{8}n}} \geq \frac{-\frac{1}{4}n}{\sqrt{\frac{15}{8}n}}\right)$$

Z is $N(0,1)$

$$\approx P\left(Z \geq -\sqrt{\frac{n}{30}}\right)$$

$$= P(Z \geq -2)$$



$$= P(Z \leq 2) = \Phi(2)$$