Math 135A, Fall 2025.
Oct. 24, 2025.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Electronic devices, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions: proper interpretation of exam questions is a part of the exam.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. You are dealt 10 cards at random from a full deck of 52 cards. Recall that the full deck contains 13 cards of each of the four suits ($\heartsuit$, $\diamondsuit$, $\spadesuit$, $\clubsuit$).

(a) Compute the probability that all cards you get are hearts ($\heartsuit$).

$$\frac{\binom{13}{10}}{\binom{52}{10}} \quad \text{or} \quad \frac{\binom{42}{3}}{\binom{52}{13}}$$

select cards you get → $\binom{52}{10}$

$\binom{52}{13}$ ← select positions of ♡s

(b) Compute the probability that all cards you get are of the same suit.

$$4 \cdot (\text{probability in (a)}) = 4 \cdot \frac{\binom{13}{10}}{\binom{52}{10}}$$

(c) Compute the probability that, among the cards you get, some two suits are represented by 3 cards each and the other two suits are represented by 2 cards each.

choose the two suits with 3 cards → $\binom{4}{2}$

$$\frac{\binom{4}{2}\binom{13}{3}^2 \cdot \binom{13}{2}^2}{\binom{52}{10}}$$

(d) Compute the probability that, among the cards you get, at least one of the three suits $\heartsuit$, $\diamondsuit$, $\spadesuit$ is missing.

$A_1 = \{\heartsuit \text{ missing}\}$, $A_2 = \{\diamondsuit \text{ missing}\}$, $A_3 = \{\spadesuit \text{ missing}\}$

IE:

$$P(A_1 \cup A_2 \cup A_3) = 3 \cdot \frac{\binom{39}{10}}{\binom{52}{10}} - 3\frac{\binom{26}{10}}{\binom{52}{10}} + \frac{\binom{13}{10}}{\binom{52}{10}}$$

$$P(A_1) = P(A_2) = P(A_3) = \frac{\binom{39}{10}}{\binom{52}{10}}$$

$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = \frac{\binom{26}{10}}{\binom{52}{10}}$$

$$P(A_1 \cap A_2 \cap A_3) = \frac{\binom{13}{10}}{\binom{52}{10}}$$

2. Roll a fair die 10 times.

(a) Compute the probability that you roll exactly four 1s.

$$\binom{10}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^6 = \frac{\binom{10}{4}5^6}{6^{10}}$$

(b) Compute the probability that among the 10 rolls there are exactly three 1s, exactly three 2s, and exactly three 3s.

$$\frac{\binom{10}{3}\binom{7}{3}\binom{4}{3}\cdot 3}{6^{10}}$$

number on remaining roll

(c) Compute the probability that among your 10 rolls there is at least one 1 and also at least one 2.

$$1 - P(\{\text{no } 1\} \cup \{\text{no } 2\})$$

$$= 1 - P(\text{no } 1) - P(\text{no } 2) + P(\text{no } 1 \text{ and no } 2)$$

$$= 1 - 2\left(\frac{5}{6}\right)^{10} + \left(\frac{4}{6}\right)^{10}$$

(d) Compute the probability that both of the following happen: all five numbers rolled on the first five rolls are different; and all five numbers on the last five rolls are different. (For example, this happens if rolls are 1234632514 but not if the rolls are 3236562514.)

Choose no. on first five rolls; order them; then do the same for last five rolls

$$\frac{\left(\binom{6}{5}\cdot 5!\right)^2}{6^{10}} = \frac{(5!)^2}{6^8}$$

3. A bag contains 2 red, 3 green, and 5 blue balls. Repeatedly perform the following steps. At each step, you select a ball at random from the bag. If the selected ball is red, you note its color, but then you put it back in the bag. If the selected ball is any other color, it is not replaced, so the number of balls is in the bag is reduced by 1 in the next step. (For example, if your first selection happens to be a green ball, then the bag contains 2 red, 2 green, and 5 blue balls after the first step is completed; on the other hand, if your first selection is a red ball, then the contents of the bag are unchanged after the first step is completed.)

(a) Compute the probability that the selected balls in first three steps are, in order, green, red, blue.

$$\frac{3}{10} \cdot \frac{2}{9} \cdot \frac{5}{9}$$

(b) Compute the probability that the selected ball on the second step is green.

$F_1 = \{\text{red on 1st step}\}$ $\quad A = \{\text{green on 2nd step}\}$

$F_2 = \{\text{green} \quad \text{—''—}\}$

$F_3 = \{\text{blue} \quad \text{—''—}\}$

i	$P(F_i)$	$P(A \mid F_i)$
1	2/10	3/10
2	3/10	2/9
3	5/10	3/9

So: $\boxed{P(A) = \frac{2}{10} \cdot \frac{3}{10} + \frac{3}{10} \cdot \frac{2}{9} + \frac{5}{10} \cdot \frac{3}{9}}$

(c) Assuming the selected ball on the second step is green, what is the conditional probability that the selected ball on the first step is red?

Bayes' formula:

$$P(F_1 \mid A) = \frac{P(F_1)\,P(A \mid F_1)}{P(A)}$$

$$= \frac{\frac{2}{10} \cdot \frac{3}{10}}{\frac{2}{10} \cdot \frac{3}{10} + \frac{3}{10} \cdot \frac{2}{9} + \frac{5}{10} \cdot \frac{3}{9}}$$

4. Alice and Bob play the following *Colors* game. In each round, they shuffle a deck of 6 cards; 2 of these cards are red and 4 are black. Alice wins if the top (i.e., first) card in the shuffled deck is red and Bob wins if the top two cards are both black. Otherwise, the game is undecided; they shuffle the deck again and play another round. Rounds are played until the *Colors* game is decided and thus one of them wins the game. Give all numerical answers as simple fractions.

(a) Compute the probability that the game is decided on the first round.

$P(\text{game decided on 1st round})$

$= P(\text{A. wins on 1st round}) + P(\text{B. wins on 1st rd.})$

$= \frac{2}{6} + \frac{4}{6} \cdot \frac{3}{5} = \frac{1}{3} + \frac{2}{5} = \underline{\underline{\frac{11}{15}}}$

(b) Compute the probability that Bob wins the game.

$$\frac{P(\text{B. wins on 1st rd.})}{P(\text{game decided on 1st rd.})} = \frac{\frac{2}{5}}{\frac{11}{15}} = \underline{\underline{\frac{6}{11}}}$$

(c) Let C be the event that the top card *on the first round* is black. Let B be the event that Bob wins the game *on the first round*. Compute $P(B \mid C)$. Are B and C independent?

$P(B|C) = P(\text{2nd card black} \mid \text{1st card black})$

$= \underline{\underline{\frac{3}{5}}}$

<u>No</u>, they are not independent, as $\underline{B \subset C}$.

(Also, $P(B|C) = \frac{3}{5} \neq \frac{2}{5} = P(B)$.)