Math 135A, Fall 2025.
Nov. 21, 2025.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Electronic devices, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions: proper interpretation of exam questions is a part of the exam. You do not need to evaluate $\Phi(z)$, for *positive* z.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Assume X is a random variable with density

$$f_X(x) = \begin{cases} c(2x+3x^2) & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

In your answers to questions below, give all numbers as simple fractions. You may use: $\int_0^1 x^n\,dx = \frac{1}{n+1}$ for $n > -1$.

(a) Compute the constant c.

$$c\int_0^1 (2x+3x^2)\,dx = 1 \qquad c(1+1) = 1$$

$$\underline{\underline{c = \frac{1}{2}}}$$

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(b) Compute EX.

$$EX = \frac{1}{2}\int_0^1 x(2x+3x^2)\,dx = \frac{1}{2}\left[2\cdot\frac{1}{3} + 3\cdot\frac{1}{4}\right]$$

$$= \underline{\underline{\frac{17}{24}}}$$

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(c) Compute the density of the random variable $Y = X^2$.

$$y \in [0,1]: \; P(Y \le y) = \frac{1}{2}\int_0^{\sqrt{y}} (2x+3x^2)\,dx$$

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$$f_Y(y) = \frac{1}{2}\cdot\frac{1}{2\sqrt{y}}\,(2\sqrt{y}+3y)$$

$$= \frac{1}{2} + \frac{3}{4}\sqrt{y} \qquad \text{if } y \in [0,1]$$

$$0 \qquad \text{otherwise}$$

2. Roll a fair die 8 times. Let X be the number of times you roll 1 and Y the number of times your roll an even number. (For example, if the rolls are 12344612, then $X = 2$ and $Y = 5$.)
(a) Write down the joint probability mass function of X and Y. (Write a formula rather than a table.)

$x, y = 0, \dots, 8$
$x + y \leq 8$

$$P(X = x, Y = y) = \frac{\binom{8}{x}\binom{8-x}{y} 3^y \cdot 2^{8-x-y}}{6^8}$$

choose the even rolls

choose roles that are odd and $\neq 1$

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(b) Are X and Y independent? No. $P(X=8) \neq 0$, $P(Y=8) \neq 0$
but $P(X=8, Y=8) = 0$,
so $P(X=8, Y=8) \neq P(X=8) \cdot P(Y=8)$

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(c) Identify (as well-known probability mass functions) the marginal probability mass functions of X and Y.

X is Binomial $(8, \frac{1}{6})$

Y is Binomial $(8, \frac{1}{2})$

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(d) Compute the conditional probability $P(X = 1 \mid Y \leq 1)$. =

$$\frac{P(X=1, Y \leq 1)}{P(Y \leq 1)} = \frac{P(X=1, Y=0) + P(X=1, Y=1)}{P(Y=0) + P(Y=1)}$$

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$$= \frac{\frac{8 \cdot 2^7}{6^8} + \frac{8 \cdot 7 \cdot 3 \cdot 2^6}{6^8}}{\left(\frac{1}{2}\right)^8 + 8 \cdot \left(\frac{1}{2}\right)^8}$$

3. Alice and Bob play the following *Three Dice* game once every day. Three fair dice are rolled. Alice wins if all three numbers rolled are different. Bob wins if all three numbers rolled are 6. (Nobody wins otherwise.)

(a) Using the relevant approximation, estimate the probability that Bob wins the *Three Dice* game at least twice in the first 432 days. (Start by computing Bob's winning probability in one instance of the *Three Dice* game. Then identify the distribution of the number Y of Bob's wins in 432 days. Note that $432 = 2 \cdot 216 = 2 \cdot 6^3$.)

$P(B.\text{ wins in one instance of the game}) = \frac{1}{6^3}$

So Y is $\text{Binomial}(2 \cdot 6^3, \frac{1}{6^3}) \approx \text{Poisson}(2)$.

$$P(Y \geq 2) = 1 - P(Y=0) - P(Y=1)$$ 12

$$\approx 1 - e^{-2} - 2e^{-2} = \underline{\underline{1 - 3e^{-2}}}$$

(b) Using the relevant approximation, estimate the probability that Alice wins the *Three Dice* game at
240 least ~~190~~ times in the first 405 days. (Start by expressing Alice's winning probability in one instance of the *Three Dice* game as a simple fraction. Then identify the distribution of the number X of Alice's wins in 405 days. Note that $405 = 5 \cdot 9 \cdot 9$ and $\overset{240}{\cancel{190}} = 5 \cdot 5 \cdot 9 + 15$.)

$P(A.\text{ wins in one instance of the game}) = \frac{6 \cdot 5 \cdot 4}{6^3} = \frac{5}{9}$ 13

So X is $\text{Binomial}(405, \frac{5}{9})$

Using Normal approximation, with Z standard Normal

$$P(X \geq 190) = P\left(\frac{X - 405 \cdot \frac{5}{9}}{\sqrt{405 \cdot \frac{5}{9} \cdot \frac{4}{9}}} \geq \boxed{\frac{240 - 405 \cdot \frac{5}{9}}{\sqrt{405 \cdot \frac{5}{9} \cdot \frac{4}{9}}}}\right)$$

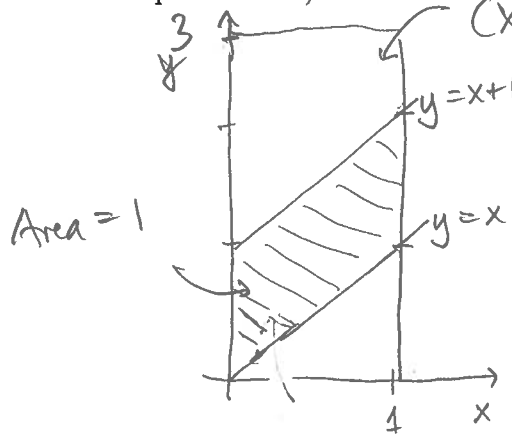
$$\| \qquad \frac{5 \cdot 5 \cdot 9 + 15 - 5 \cdot 9 \cdot \cancel{9} \cdot \frac{5}{\cancel{9}}}{\sqrt{5 \cdot 9 \cdot \cancel{9} \cdot \frac{5}{\cancel{9}} \cdot \frac{4}{\cancel{9}}}} = \frac{15}{10}$$

$$\approx P(Z \geq 1.5) = 1 - P(Z \leq 1.5)$$

$$= \underline{\underline{1 - \Phi(1.5)}}$$

4. Alice and Bob play the following *Points* game once every day, starting on day 1. Alice chooses a random number X uniformly on $[0, 1]$ and Bob independently chooses a random number Y uniformly on $[0, 3]$. Alice wins if $X \le Y \le X + 1$.

(a) Compute the probability that Alice wins the game. (Draw the picture and give the result as as simple fraction.)



(X,Y) is a random pt. in this rectangle

$$P(X \le Y \le X+1) = \frac{(\text{Area of Shaded set})}{3}$$

$$= \underline{\underline{\frac{1}{3}}}$$

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(b) Let N be the number of days needed for Alice to win the *Points* game for the first time. (That is: if Alice wins on day 1, then $N = 1$; if Alice does not win on day 1 and wins on day 2, then $N = 2$, etc.) Identify the probability mass function of N and compute EN.

N is Geometric$(1/3)$, so $\underline{\underline{EN = 3}}$.

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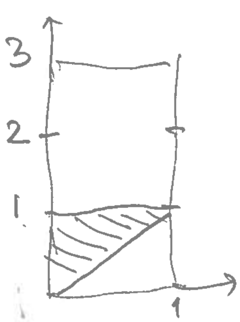
(c) You meet Alice later on day 1 and she tells you she won the *Points* game on that day. What is now the probability that Bob chose a number in $[0, 1]$ on that day?



$$P(Y \in [0,1] \mid X \le Y \le X+1)$$

$$= \frac{P(Y \in [0,1],\ X \le Y \le X+1)}{P(X \le Y \le X+1)}$$

$$= \frac{\frac{1}{3}(\text{Area of shaded set})}{1/3} = \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{3}} = \underline{\underline{\frac{1}{2}}}$$

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