Lecture 4: Combinations of sets

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Combinations of sets

Assume we have a set S of n objects. Let $\binom{n}{r}$, "n choose r" be the number of subsets of S with precisely r elements.
For example, $\binom{n}{0} = \binom{n}{n} = 1$, $\binom{n}{1} = n$. We also assume $\binom{n}{r} = 0$ when $r \notin [0, n]$.
For reasons we will explain later, $\binom{n}{r}$ is also called a *binomial coefficient*.

Combinations of sets

Theorem

For $n \geq 1$, $0 \leq r \leq n$,

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}.$$

Proof.

Define an eqivalence relation on the set of r-permutations of an n-elements set S by considering two r-permutations equivalent if they are two orderings of the same subset of S. Then every equivalence class has the same size $r!$, and the number of equivalence classes is $\binom{n}{r}$. Now we apply the division principle. □

Two immediate properties of binomial coefficients

Choosing a set is the same is choosing its complement:

$$\binom{n}{r} = \binom{n}{n-r}$$

We can choose a subset by choosing its size first:

$$\sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

Combinations of sets: examples

Example 4.1. A basketball team has 12 players. How many selections of start-up five players are possible (disregarding the playing positions)?

Answer: $\binom{12}{5} = 792$.

Combinations of sets: examples

Example 4.2. How many five-card poker hands?

Answer: $\binom{52}{5} = 2\,598\,980$.

How many five-card poker hands with a *three of a kind*? (Three denominations (or values), with 3 cards of the same denomination, e.g. $K\heartsuit$, $K\diamondsuit$, $K\spadesuit$, $Q\heartsuit$, $3\clubsuit$.)

Combinations of sets: examples

How many five-card poker hands with a *three of a kind*?

Answer: Choose the value for the triple, then other two values, then the 3 cards of the triple for the chosen value, then each of the two cards with given values.
We get:

$$13 \cdot \binom{12}{2} \cdot \binom{4}{3} \cdot 4^2 = 54\,912.$$

Example 4.3. You want to order 15 bottles of wine on a shelf: 5 indistinguishable bottles of red wine and 10 indistinguishable bottles of white wine. In how many ways can you do this:
(a) with no restriction;
(b) All red bottles and all white wine bottles are next to each other;
(c) no red wine bottle is next to another red wine bottle?

You want to order 15 bottles of wine on a shelf: 5 indistinguishable bottles of red wine and 10 indistinguishable bottles of white wine. In how many ways can you do this:
(a) with no restriction?

Answer to (a): Choose the positions of red wine bottles to get $\binom{15}{5} = 3003$.

Combinations of sets: examples

You want to order 15 bottles of wine on a shelf: 5 indistinguishable bottles of red wine and 10 indistinguishable bottles of white wine. In how many ways can you do this:
(b) All red bottles and all white wine bottles are next to each other?

Answer to (b): 2, as the only choice you have is the order of the two blocks.

Combinations of sets: examples

You want to order 15 bottles of wine on a shelf: 5 indistinguishable bottles of red wine and 10 indistinguishable bottles of white wine. In how many ways can you do this:
(c) no red wine bottle is next to another red wine bottle?

Answer to (c). Any position of a red wine bottle, except the rightmost one, excludes the position to its right. So 4 of the red wine bottles occupy two adjacent slots, eliminating 4 positions. Answer

$$\binom{15-4}{5} = \binom{11}{5} = 462.$$

We solve a previous example again in a slightly different fashion.

Example 4.4. A group of 20 (distinguishable!) Scandinavians includes 6 Swedes, and 14 Norwegians. In how many ways can they be seated on 20 chairs (a) in a row and (b) around the table so that no two Swedes sit together.

Answer to (a). Choose the seats for Swedes, then sit the Swedes, then the Norwegians: $\binom{15}{6} \cdot 6! \cdot 14!$.

Answer to (b). Sit Magnus the Norwegian on the prescribed seat, choose the seats for Swedes from among the 19 remaining seats, then sit the Swedes, then the remaining Norwegians: $\binom{14}{6} \cdot 6! \cdot 13!$.

Combinations of sets: examples

Example 4.5. How many 7-digit numbers have:
(a) Exactly 3 0s?
(b) Exactly 4 digits in $[1, 6]$?
(c) At least two digits in $[1, 6]$?

Combinations of sets: examples

How many 7-digit numbers have:
(a) Exactly 3 0s?

Answer to (a): Choose the positions for the 0s, then fill with other numbers to get $\binom{6}{3} \cdot 9^4$.

Combinations of sets: examples

How many 7-digit numbers have:
(b) Exactly 4 digits in $[1,6]$?

Answer to (b): The first digit could be in $[7,9]$ or in $[1,6]$. In each case, choose the position of the digits in $[1,6]$ among the remaining digits, then fill in to get

$$3 \cdot \binom{6}{4} \cdot 6^4 \cdot 4^2 + 6 \cdot \binom{6}{3} \cdot 6^3 \cdot 4^3.$$

Combinations of sets: examples

How many 7-digit numbers have:
(c) At least two digits in $[1, 6]$?

Answer to (c): Compute the number of numbers with zero or one such digit instead. In case of one such digit, it can be the first digit or not. We get:

$$9 \cdot 10^6 - 3 \cdot 4^6 - 6 \cdot 4^6 - 3 \cdot 6 \cdot 6 \cdot 4^5.$$