Lecture 17: Examples of linear recursion

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General linear recursion

A sequence f_n, $n = 0, 1, 2, \ldots$ satisfies a *k'th order linear recursion* if

$$\text{(R)} \qquad f_n = a_1 f_{n-1} + a_2 f_{n-2} + \ldots + a_k f_{n-k}$$

for $n \geq k$, for some constants $a_1, \ldots, a_k \in \mathbb{R}$, with $a_k \neq 0$. The *characteristic equation* of this recursion is

$$\text{(CE)} \qquad x^k - a_1 x^{k-1} - a_2 x^{k-2} - \ldots - a_k = 0$$

and its solutions are *characteristic roots*. Assume that $q_1, \ldots, q_\ell$, $\ell \leq k$, are the disctinct characteristic roots of respective multiplicities $s_1, \ldots, s_\ell$, so that $s_1 + \ldots + s_\ell = k$. Take a sequence h_n of the form

$$\text{(G)} \qquad \begin{aligned} h_n = {} & (c_{11} + c_{12} n + \ldots + c_{1 s_1} n^{s_1 - 1}) q_1^n \\ & \ldots \\ & + (c_{\ell 1} + c_{\ell 2} n + \ldots + c_{\ell s_\ell} n^{s_\ell - 1}) q_\ell^n, \end{aligned}$$

where the c's are k real constants.

General linear recursion

Theorem

For any selection of the constants c, the sequence h_n given by (G) is a solution of the recursion (R). Moreover, there is a unique choice of constants c for which $h_n = f_n$ for $n = 0, \ldots, k-1$, and for this choice $h_n = f_n$ for all $n \geq 0$.

For this reason, the sequence h_n given by (G) is called the *general solution* of the recursion (R). Note that some of the q's may be complex!

Solving linear recursion: examples

Example 17.1. Let b_n be the number of ways in which $1 \times n$ chessboard can be perfectly covered using red, white and blue dominoes, and yellow monominoes (each in infinite supply).

(a) Write the recursive equation and determine the initial conditions.
(b) Solve the recursion.
(c) Determine $\lim b_{n+1}/b_n$.
(d) Now redefine b_n with the additional condition that two monominoes cannot be next to each other. Determine the limit in (c) to three decimal places.

Solving linear recursion: examples

Let b_n be the number of ways in which $1 \times n$ chessboard can be covered using red, white and blue dominoes, and yellow monominoes (each in infinite supply).
(a) Write the recursive equation and determine the initial conditions.

We have

$$b_n = b_{n-1} + 3b_{n-2}$$

and $b_1 = 1$, $b_2 = 4$, or better $b_0 = 1$.

Solving linear recursion: examples

(b) Solve the recursion.
We have

$$b_n = b_{n-1} + 3b_{n-2}, \quad b_0 = b_1 = 1.$$

The characteristic equation $x^2 - x - 3 = 0$ has the solution

$$x = \frac{1 \pm \sqrt{13}}{2},$$

and the general solution is

$$b_n = c_1 \left(\frac{1 + \sqrt{13}}{2} \right)^n + c_2 \left(\frac{1 - \sqrt{13}}{2} \right)^n .$$

Solving linear recursion: examples

By initial conditions:

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 \frac{1+\sqrt{13}}{2} + c_2 \frac{1-\sqrt{13}}{2} &= 1 \end{aligned}$$

So,

$$c_1 \cdot \sqrt{13} = 1 - \frac{1-\sqrt{13}}{2} = \frac{1+\sqrt{13}}{2}$$

and

$$c_1 = \frac{1+\sqrt{13}}{2\sqrt{13}}, \quad c_2 = \frac{-1+\sqrt{13}}{2\sqrt{13}}.$$

Answer:

$$b_n = \frac{1+\sqrt{13}}{2\sqrt{13}} \left(\frac{1+\sqrt{13}}{2} \right)^n + \frac{-1+\sqrt{13}}{2\sqrt{13}} \left(\frac{1-\sqrt{13}}{2} \right)^n$$

The answer to (c):

$$\lim \frac{b_{n+1}}{b_n} = \frac{1+\sqrt{13}}{2} \approx 2.30.$$

Solving linear recursion: examples

Let b_n be the number of ways in which $1 \times n$ chessboard can be covered using red, white and blue dominoes, and yellow monominoes (each in infinite supply), so that
(d) two monominoes cannot be next to each other.

Now you either begin with a domino, or with a monomino and then a domino. So the recursion now is

$$b_n = 3b_{n-2} + 3b_{n-3}$$

and is of the third order. We need three initial conditions: $b_0 = 1$, $b_1 = 1$, $b_2 = 3$.

Solving linear recursion: examples

Determine $\lim b_{n+1}/b_n$ to three decimal places.

The characteristic equation $x^3 - 3x - 3 = 0$ has roots, according to MATLAB:

$$2.1038, \ -1.0519 \pm 0.5652i$$

with absolute values

$$2.1038, \ 1.1941, \ 1.1941$$

and the coefficient of the largest root must be nonzero (by numerical computation, or by $b_n \geq 3b_{n-2}$ so $b_n \geq (\sqrt{3})^n$), so

$$\lim \frac{b_{n+1}}{b_n} \approx 2.104.$$

Example 17.2. A bug walks on three points a, b, c. At each step, it may either stay at the same point, or move as follows: from a it may move to either b or c, from b it may move to a and from c it may also move to a. (For example, if the bug is at b, it can be either at b or at a at the next step.) Assume the bug starts at a.

(a) In how many ways can the bug make n steps?
(b) What about if it needs to end up at a?

Solving linear recursion: examples

For (a), let a_n be the number of ways starting at a and b_n be the number of ways starting at b. Then we have

$$a_n = a_{n-1} + 2b_{n-1}$$
$$b_n = a_{n-1} + b_{n-1}$$

As

$$a_{n-1} = a_{n-2} + 2b_{n-2}$$
$$b_{n-1} = a_{n-2} + b_{n-2}$$

we get

$$2b_{n-2} = a_{n-1} - a_{n-2}$$

and then

$$2b_{n-1} = 2a_{n-2} + 2b_{n-2} = 2a_{n-2} + a_{n-1} - a_{n-2} = a_{n-1} + a_{n-2}.$$

It follows that a_n satisfies the second order recursion

$$a_n = 2a_{n-1} + a_{n-2}$$

with initial conditions $a_1 = 3$, $a_2 = 7$, or better $a_0 = 1$.

The characteristic equation $x^2 - 2x - 1 = 0$ has solutions

$$\frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2},$$

and so

$$a_n = c_1 \left(1 + \sqrt{2}\right)^n + c_2 \left(1 - \sqrt{2}\right)^n$$

Solving linear recursion: examples

By initial conditions:

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1(1+\sqrt{2}) + c_2(1-\sqrt{2}) &= 3 \end{aligned}$$

and so

$$c_1 = \frac{1+\sqrt{2}}{2}, \quad c_2 = \frac{1-\sqrt{2}}{2}.$$

Solving linear recursion: examples

(b) What about if it needs to end up at a?

Again let a_n be the number of ways starting at a (and ending in a) and b_n be the number of ways starting at b (and ending in a) . The recursion is exactly the same as before, but the initial conditions change: $a_0 = 1$ and $a_1 = 1$.

(Note that here $b_0 = 0$, so it is not always true that the 0'th term is 1!)

Example 17.3. Assume that the sequence a_n satisfies a linear recursive equation of order 5 with characteristic equation $(x-2)^5 = 0$. Assume $a_0 = a_1 = a_2 = a_3 = a_4 = 1$.

(a) What is a_5?

(b) Determine the three positive constants α , β, γ so that, as $n \to \infty$,

$$\lim \frac{a_n}{\alpha n^\beta \gamma^n} \to 1.$$

Solving linear recursion: examples

For (a), the binomial theorem for $(x-2)^5 = ((-2)+x)^5$ gives

$$(x-2)^5 = \sum_{i=0}^{5} \binom{5}{i} (-1)^i 2^i x^{5-i} = x^5 - \sum_{i=1}^{5} \binom{5}{i} (-1)^{i+1} 2^i x^{5-i},$$

the recursion of order 5 is

$$a_n = \sum_{i=1}^{5} \binom{5}{i} (-1)^{i+1} 2^i a_{n-i}.$$

Therefore

$$\begin{aligned} a_5 &= \sum_{i=1}^{5} \binom{5}{i} (-1)^{i+1} 2^i = -\sum_{i=1}^{5} \binom{5}{i} (-2)^i \\ &= -\left(\sum_{i=0}^{5} \binom{5}{i} (-2)^i - 1 \right) = -((1-2)^5 - 1) = 2. \end{aligned}$$

Solving linear recursion: examples

Assume that the sequence a_n satisfies a linear recursive equation of order 5 with characteristic equation $(x-2)^5 = 0$. Assume $a_0 = a_1 = a_2 = a_3 = a_4 = 1$.
(b) Determine the three positive constants α , β, γ so that, as $n \to \infty$,

$$\lim \frac{a_n}{\alpha n^\beta \gamma^n} \to 1.$$

The general solution of the recursion is

$$(c_0 + c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4) 2^n$$

so if we determine c_4 and it is nonzero, we are done.

Solving linear recursion: examples

For this, it is convenient to change the basis:

$$\left(d_0 + d_1 n + d_2 \binom{n}{2} + d_3 \binom{n}{3} + d_4 \binom{n}{4}\right) 2^n$$

and observe that $c_4 = d_4/4!$ (although other constants do not relate so easily).

The d's are much easier to compute:
$a_0 = 1 \Longrightarrow d_0 = 1$
$a_1 = 1 \Longrightarrow d_0 + d_1 = 1/2 \Longrightarrow d_1 = -1/2$
$a_2 = 1 \Longrightarrow d_0 + 2d_1 + d_2 = 1/4 \Longrightarrow d_2 = 1/4$
$a_3 = 1 \Longrightarrow d_0 + 3d_1 + 3d_2 + d_3 = 1/8 \Longrightarrow d_3 = -1/8$
$a_4 = 1 \Longrightarrow d_0 + 4d_1 + 6d_2 + 4d_3 + d_4 = 1/16 \Longrightarrow d_4 = 1/16$

The answer is

$$a_n \sim \frac{1}{384} n^4 2^n.$$