Math 16C, Winter 1997.
Mar. 20, 1997

FINAL EXAM

NAME(print): KEY

NAME(sign):

ID#:

Instructions: Each of the eight problems is worth 25 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor in determining credit. Calculators, books or notes are not allowed, with the exception of one index card, containing Maclaurin series.

Make sure that you have a total of 9 pages (including this one) with 8 problems. Read through the entire exam before beginning to work.

1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

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1. For each of the following two differential equations, find a particular solution which passes through the point (0,1).

(a) $y' = (y+1)\cdot \sin x$

$$\frac{dy}{dx} = (y+1)\sin x$$

$$\int \frac{dy}{y+1} = \int \sin x \, dx$$

$$\ln(y+1) = -\cos x + C$$

$$y+1 = Ce^{-\cos x}$$

$$\underline{y = Ce^{-\cos x} - 1} \quad \text{gen. sol.}$$

$$1 = Ce^{-1} - 1$$

$$C = 2e$$

$$\boxed{y = 2e\, e^{-\cos x} - 1}$$

(b) $y' + y = x$

$$P = 1 \qquad u = e^{\int P\,dx} = e^{x}$$

$$y = e^{-x}\int xe^{x}\,dx = \underline{x - 1 + Ce^{-x}} \quad \text{gen. sol.}$$

$$\int xe^{x}\,dx = xe^{x} - \int e^{x}\,dx = xe^{x} - e^{x} + C$$

$$u = x \qquad dv = e^{x}\,dx$$

$$du = dx \qquad v = e^{x}$$

Plugging in $x=0,\ y=1$

$$1 = -1 + C, \quad \underline{C = 2}, \quad \boxed{y = x - 1 + 2e^{-x}}$$

2. A company sells a certain product. Let x be the amount of sales, which changes with time t. Initially, the company starts selling the product, so $x = 0$ when $t = 0$. After one year, the company has sold 5,000 units of the product. The marketing department estimates that the company can sell at most 10,000 units, and that the rate of change in sales is proportional to the difference between between the maximum sales and the current sales.

(a) Write down the differential equation for x, and find its general solution. Unit = 10000.
(You can make your comp. easier by choosing units)

$$\frac{dx}{dt} = k(1-x)$$

$$\frac{dx}{1-x} = k\,dt$$

$$-\ln(1-x) = kt + C$$

$$1-x = Ce^{-kt}$$

$$\boxed{x = 1 - Ce^{-kt}}$$

(b) Find the amount of sales after 2 years.

$t=0,\ x=0:\quad 0 = 1 - C,\quad C = 1$

$t=1,\ x=0.5:\quad 0.5 = 1 - e^{-kt}$

$$e^{-k} = 0.5$$

$$-k = \ln\tfrac{1}{2}$$

$$k = \ln 2$$

$\underline{\underline{x}} = 1 - e^{-(\ln 2)\cdot t} = \underline{\underline{1 - 2^{-t}}}$ part. sol.

$t=2:\quad x = \frac{3}{4}\quad \boxed{7.500}$ i.e. amount of sales after

3. Find and classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$.

$f_x = 3x^2 - 3y = 3(x^2 - y) = 0$ $\quad y = x^2$

$f_y = 3y^2 - 3x = 3(y^2 - x) = 0$

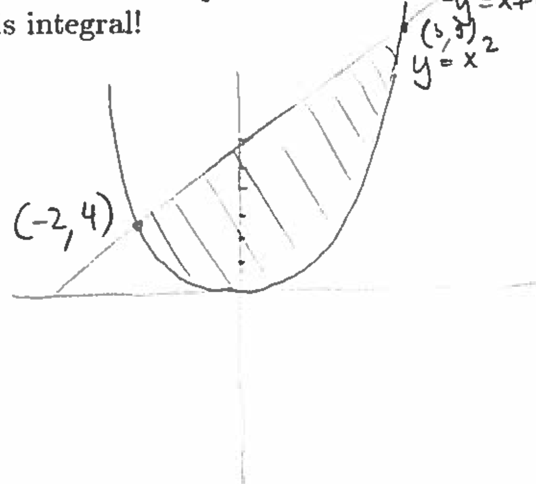
$y = x^2 \qquad y^2 - x = 0, \quad x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0, \; x = 1$

Critical points:	$(0, 0)$,	$(1, 1)$.
$f_{xx} = 6x$	0	6
$f_{yy} = 6y$	0	6
$f_{xy} = -3$	-3	-3
$d = f_{xx} f_{yy} - f_{xy}^2$	-9	$36 - 9 = 25$
	$\uparrow$ saddle	$\uparrow$ local min

4. Assume that a solid is bounded above by the plane $z = 20 - 2x - y$, ~~and~~ while its base is given by the region in the xy plane which lies between the graphs of $y = x^2$ and $y = x + 6$. Set up the double integral for the volume of this solid. DO NOT evaluate this integral!



$$x^2 = x + 6$$

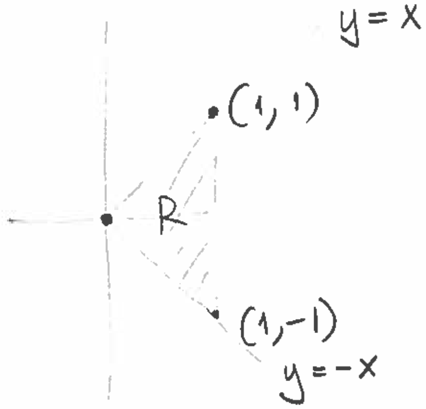
$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = 3, \quad x = -2$$

$$\int_{-2}^{3} \left[\int_{x^2}^{x+6} (20 - 2x - y)\, dy \right] dx$$

5. Compute the average value of the function $f(x, y) = 3x^2y + 3y^2$ on the triangle with vertices (0,0), (1,1) and (1,-1).



area (R) = 1

$$\int_0^1 \int_{-x}^{x} (3x^2y + 3y^2)\, dy \quad dx$$

$$= \int_0^1 \left[3x^2 \frac{y^2}{2} + y^3 \right]_{y=-x}^{y=x} dx$$

$$= \int_0^1 \left(\cancel{3x^2 \frac{x^2}{2}} + x^3 - \cancel{3x^2 \cdot \frac{x^2}{2}} + x^3 \right) dx$$

$$= \int_0^1 2x^3\, dx = \underline{\underline{\frac{1}{2}}}$$

6. For each of the following series, determine whether it converges or diverges. State *clearly* your reasoning.

(a) $\sum_{n=1}^{\infty} \frac{3^n}{3^{n+1}+2^n}$

$$a_n = \frac{3^n}{3^{n+1}+2^n} = \frac{1}{3+\left(\frac{2}{3}\right)^n} \longrightarrow \frac{1}{3}$$

<u>diverges</u>

(n'th term test)

(b) $\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{3^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)\cdot 2^{n+1}}{3^{n+1}}}{\frac{n \cdot 2^n}{3^n}} = \frac{(n+1)\, 2^{\cancel{n+1}} \cdot 3^{\cancel{n}}}{n\; 3^{\cancel{n+1}} \cdot \cancel{2^n}} = \frac{2}{3}\,\frac{n+1}{n} \xrightarrow[n\to\infty]{} \frac{2}{3}$$

<u>converges</u> (ratio test)

(c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$p = 1/2$, p-series, <u>diverges</u>

(d) $\sum_{n=0}^{\infty} \frac{5^{n/2}}{3^n} = \sum_{n=0}^{\infty} \left(\frac{\sqrt{5}}{3}\right)^n$

Geometric series $r = \frac{\sqrt{5}}{3} < 1$. <u>converges</u>

7. (a) Write down the Maclaurin series for $f(x) = x \cdot \ln(2+5x)$. For which x does this series converge?

$$x \ln(2+5x) = x \ln\left(2\cdot\left(1+\tfrac{5}{2}x\right)\right)$$

$$= x \ln 2 + x \ln\left(1+\tfrac{5}{2}x\right)$$

$$= x \ln 2 + x\left(\tfrac{5}{2}x - \tfrac{1}{2}\left(\tfrac{5}{2}x\right)^2 + \tfrac{1}{3}\left(\tfrac{5}{2}x\right)^3 - \cdots\right)$$

$$= \ln 2 \cdot x + \tfrac{5}{2}x^2 - \tfrac{1}{2}\left(\tfrac{5}{2}\right)^2 x^3 + \tfrac{1}{3}\left(\tfrac{5}{2}\right)^3 x^4 - \cdots$$

Converges for $\left|\frac{5}{2}x\right| < 1$, that is $|x| < \frac{2}{5}$.

(b) Find a closed form expression for the power series

$$2x - \frac{2^3x^3}{2!} + \frac{2^5x^5}{4!} - \frac{2^7x^7}{6!} + \ldots$$

For which x does this series converge?

$$= 2x\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right)$$

$$= 2x \cos(2x)$$

converges for every x.

8. Use the ~~fourth~~ sixth degree Taylor polynomial to estimate

$$\int_0^{0.5} (1+x)\sin(x^2)\,dx$$

$$(1+x)\sin x^2 = (1+x)\left(x^2 - \frac{x^6}{3!} + \cdots\right)$$

$$= x^2 + x^3 - \frac{x^6}{6} + \cdots$$

$$\int_0^{0.5} (1+x)\sin(x^2)\,dx \approx \int_0^{0.5} \left(x^2 + x^3 - \frac{x^6}{6}\right) dx$$

$$= \cancel{x}\frac{(0.5)^3}{3} + \frac{(0.5)^4}{4} - \frac{(0.5)^7}{42}$$