MAT 16C, Winter 2020

Lecture notes for last week of classes

Note. Final version (March 12, 2020).

We will always make the default choice for the Taylor series to be centered at $c = 0$. With this convention, Taylor series and Maclaurin series have the same meaning and can be used interchangeably. We now list the Taylor series to remember. The first is the series for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1}$$

which converges for all real numbers x, that is, its radius of convergence is $R = \infty$. Next are the series for $\sin x$ and $\cos x$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \tag{3}$$

which both converge for all real numbers x, which again means that each has radius of convergence is $R = \infty$. Next is the geometric series

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \tag{4}$$

which converges for $|x| < 1$; its radius of convergence is $R = 1$. We can plug in $-x$ in place of x to get

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \tag{5}$$

with the same radius of convergence $R = 1$, as $|-x| = |x|$. We can compute the antiderives of both sides (i.e. we integrate both sides) to get

$$\ln(1+x) + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

and then plug in $x = 0$ to get $C = 0$, so that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \tag{6}$$

which converges for $|x| < 1$. The final series is called *binomial* series, valid for any power k:

$$(1+x)^k = 1 + kx - \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots, \tag{7}$$

which again converges for $|x| < 1$.

The numbered series are the ones we will use.

The *n-th degree Taylor polynomial* $P_n(x)$ is the part of the Taylor series (again, by default centered at 0) up to and including the nth power of x. This is where Taylor series can be useful: the Taylor polynomial, especially one of high degree is a good approximation for the function if x is small.

Example. Find the 3rd degree Taylor polynomial $P_3(x)$ for $f(x) = e^x$.

As
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$
we conclude that
$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Example. Find the 3rd degree Taylor polynomial $P_3(x)$ for $f(x) = \sqrt{1+x}$.

We use the binomial series with $k = 1/2$ to get

$$\begin{aligned}
\sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-1)}{3!}x^3 + \cdots \\
&= 1 + \frac{1}{2}x - \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-1)}{3!}x^3 + \cdots \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots
\end{aligned}$$

and so
$$P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

Example. Compute
$$\frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots = \sum_{n=2}^{\infty} \frac{3^n}{n!}$$

We rewrite
$$\sum_{n=2}^{\infty} \frac{3^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} - 1 - \frac{3^1}{1!} = e^3 - 1 - 3 = e^3 - 4.$$

Example. Find the Taylor series for $\ln(3x+4)$ and its radius of convergence.

We rewrite
$$\ln(3x+4) = \ln\left(4\left(1 + \frac{3}{4}x\right)\right) = \ln 4 + \ln\left(1 + \frac{3}{4}x\right)$$
and then replace x by $\frac{3}{4}x$ in the series for $\ln(1+x)$ to get

$$\ln(3x+4) = \ln 4 + \ln\left(1 + \frac{3}{4}x\right) = \ln 4 + \frac{3}{4}x - \frac{1}{2}\left(\frac{3}{4}\right)^2 x^2 + \frac{1}{3}\left(\frac{3}{4}\right)^3 x^3 - \frac{1}{4}\left(\frac{3}{4}\right)^4 x^4 + \cdots$$

As we replaced x in the series for $\ln(1+x)$ by $\frac{3}{4}x$, this series converges for $|\frac{3}{4}x| < 1$, that is, for $|x| < \frac{4}{3}$. The radius of convergence is $R = \frac{4}{3}$.

Example. Find the sum of

$$3x + \frac{3^2x^2}{1!} + \frac{3^3x^3}{2!} + \frac{3^4x^4}{3!} + \cdots$$

For which x does it converge?

We can rewrite the above as follows

$$3x\left(1 + \frac{3x}{1!} + \frac{3^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots\right)$$

and we see that the series in parenthesis is exactly the Taylor series for e^{3x}, and so the answer is $3xe^{3x}$. As there is no restriction of convergence of the series for e^x, there is also no restriction on the series for e^{3x}, and the series converges for all x.

Example. Find the series for $(1-x)^{-2}$.

We can use the binomial series, but even easier way is to observe that $\frac{d}{dx}(1-x)^{-1} = (1-x)^{-2}$ and so by differentiating the geometric series we get

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Example. Find the 6th degree Taylor polynomial for $f(x) = e^{x^2}$.

We substitute x^2 in place of x in the series for e^x to get

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

and so

$$P_6(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6}$$

Example. Use the 4th degree Taylor polynomial to estimate

$$\int_0^{0.1} e^{x^2}\,dx$$

We estimate

$$\int_0^{0.1} e^{x^2}\,dx \approx \int_0^{0.1} P_4(x)\,dx,$$

and we know from the previous example that $P_4(x) = 1 + x^2 + \frac{x^4}{2}$, so that

$$\int_0^{0.1} P_4(x)\,dx = \int_0^{0.1}\left(1 + x^2 + \frac{x^4}{2}\right)dx = \left(x + \frac{x^3}{3} + \frac{x^5}{10}\right)\Bigg|_{x=0}^{x=0.1} = 0.1 + \frac{(0.1)^3}{3} + \frac{(0.1)^5}{10}.$$

Example. Use the 4th degree Taylor polynomial to estimate

$$\int_0^{0.5} x^2 \ln(3+2x)\,dx$$

We write

$$\ln(2+3x) = \ln 2 + \ln\left(1 + \frac{3}{2}x\right) = \ln 2 + \frac{3}{2}x - \frac{1}{2}\left(\frac{3}{2}x\right)^2 + \cdots = \ln 2 + \frac{3}{2}x - \frac{9}{8}x^2 + \cdots$$

and so

$$x^2 \ln(2+3x) = \ln 2 \cdot x^2 + \frac{3}{2}x^2 - \frac{9}{8}x^4 + \cdots$$

and

$$P_4(x) = x^2 \ln(2+3x) = \ln 2 \cdot x^2 + \frac{3}{2}x^2 - \frac{9}{8}x^4.$$

It follows that

$$\begin{aligned}
\int_0^{0.1} P_4(x)\,dx &= \int_0^{0.5} \left(\ln 2 \cdot x^2 + \frac{3}{2}x^2 - \frac{9}{8}x^4\right)\,dx \\
&= \left(\ln 2 \cdot \frac{1}{3}x^3 + \frac{3}{2}\cdot\frac{1}{4}x^4 - \frac{9}{8}\cdot\frac{1}{5}x^5\right)\Bigg|_{x=0}^{x=0.5} \\
&= \ln 2 \cdot \frac{1}{3}\cdot\frac{1}{8} + \frac{3}{2}\cdot\frac{1}{4}\cdot\frac{1}{16} - \frac{9}{8}\cdot\frac{1}{5}\cdot\frac{1}{32}.
\end{aligned}$$

Example. Use the 5th degree Taylor polynomial to estimate

$$\int_0^{0.2} \sqrt{x^4+1}\sin(x)\,dx$$

We know that (as $3! = 6$ and $5! = 120$)

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots$$

and

$$\sqrt{1+x^4} = 1 + \frac{1}{2}x^4 + \cdots$$

In this case we need to multiply the two expressions, omitting all powers of degree at least 6. This may in general be a bit tedious but in this case it is not a big problem. We get

$$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{2}x^5 = x - \frac{1}{6}x^3 + \frac{61}{120}x^5.$$

It follows that

$$\begin{aligned}
\int_0^{0.2} P_5(x)\,dx &= \left(\frac{1}{2}\cdot x^2 - \frac{1}{6}\cdot\frac{1}{4}\cdot x^4 + \frac{61}{120}\cdot\frac{1}{6}\cdot x^6\right)\Bigg|_{x=0}^{x=0.2} \\
&= \frac{1}{2}\cdot(0.2)^2 - \frac{1}{6}\cdot\frac{1}{4}\cdot(0.2)^4 + \frac{61}{120}\cdot\frac{1}{6}\cdot(0.2)^6.
\end{aligned}$$