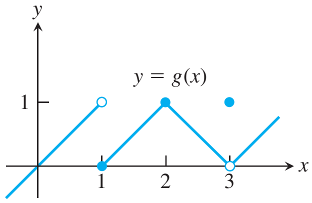
EXERCISES 2.2

Limits from Graphs

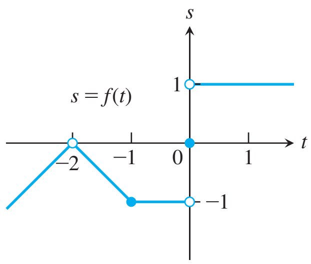
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{x\to 1} g(x)$ b. $\lim_{x\to 2} g(x)$ c. $\lim_{x\to 3} g(x)$ d. $\lim_{x\to 2.5} g(x)$



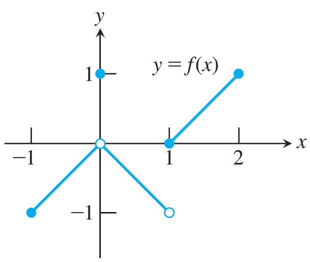
2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{t\to -2} f(t)$ b. $\lim_{t\to -1} f(t)$ c. $\lim_{t\to 0} f(t)$ d. $\lim_{t\to -0.5} f(t)$



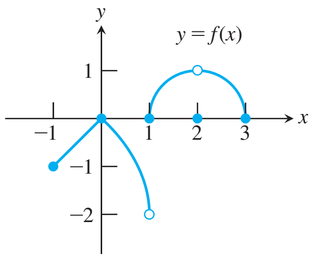
3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

   a. $\lim_{x\to 0} f(x)$ exists.
   b. $\lim_{x\to 0} f(x) = 0$
   c. $\lim_{x\to 0} f(x) = 1$
   d. $\lim_{x\to 1} f(x) = 1$
   e. $\lim_{x\to 1} f(x) = 0$
   f. $\lim_{x\to c} f(x)$ exists at every point c in $(-1, 1)$.
   g. $\lim_{x\to 1} f(x)$ does not exist.
   h. $f(0) = 0$
   i. $f(0) = 1$
   j. $f(1) = 0$
   k. $f(1) = -1$



4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

   a. $\lim_{x\to 2} f(x)$ does not exist.
   b. $\lim_{x\to 2} f(x) = 2$
   c. $\lim_{x\to 1} f(x)$ does not exist.
   d. $\lim_{x\to c} f(x)$ exists at every point c in $(-1, 1)$.
   e. $\lim_{x\to c} f(x)$ exists at every point c in $(1, 3)$.
   f. $f(1) = 0$
   g. $f(1) = -2$
   h. $f(2) = 0$
   i. $f(2) = 1$



Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim_{x\to 0} \dfrac{x}{|x|}$

6. $\lim_{x\to 1} \dfrac{1}{x-1}$

7. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x\to c} f(x)$? Give reasons for your answer.

8. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

9. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x\to -3} (x^2 - 13)$

12. $\lim_{x\to 2} (-x^2 + 5x - 2)$

13. $\lim_{t\to 6} 8(t-5)(t-7)$

14. $\lim_{x\to -2} (x^3 - 2x^2 + 4x + 8)$

15. $\lim_{x\to 2} \dfrac{2x+5}{11-x^3}$

16. $\lim_{s\to 2/3} (8-3s)(2s-1)$

17. $\lim_{x\to -1/2} 4x(3x+4)^2$

18. $\lim_{y\to 2} \dfrac{y+2}{y^2+5y+6}$

19. $\lim_{y\to -3} (5-y)^{4/3}$

20. $\lim_{z\to 4} \sqrt{z^2-10}$

21. $\lim_{h\to 0} \dfrac{3}{\sqrt{3h+1}+1}$

22. $\lim_{h\to 0} \dfrac{\sqrt{5h+4}-2}{h}$

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x\to 5} \dfrac{x-5}{x^2-25}$

24. $\lim_{x\to -3} \dfrac{x+3}{x^2+4x+3}$

25. $\lim_{x\to -5} \dfrac{x^2+3x-10}{x+5}$

26. $\lim_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

27. $\lim_{t\to 1} \dfrac{t^2+t-2}{t^2-1}$

28. $\lim_{t\to -1} \dfrac{t^2+3t+2}{t^2-t-2}$

29. $\lim_{x\to -2} \dfrac{-2x-4}{x^3+2x^2}$

30. $\lim_{y\to 0} \dfrac{5y^3+8y^2}{3y^4-16y^2}$

31. $\lim_{x\to1} \frac{x^{-1}-1}{x-1}$

32. $\lim_{x\to0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$

33. $\lim_{u\to1} \frac{u^4-1}{u^3-1}$

34. $\lim_{v\to2} \frac{v^3-8}{v^4-16}$

35. $\lim_{x\to9} \frac{\sqrt{x}-3}{x-9}$

36. $\lim_{x\to4} \frac{4x-x^2}{2-\sqrt{x}}$

37. $\lim_{x\to1} \frac{x-1}{\sqrt{x+3}-2}$

38. $\lim_{x\to-1} \frac{\sqrt{x^2+8}-3}{x+1}$

39. $\lim_{x\to2} \frac{\sqrt{x^2+12}-4}{x-2}$

40. $\lim_{x\to-2} \frac{x+2}{\sqrt{x^2+5}-3}$

41. $\lim_{x\to-3} \frac{2-\sqrt{x^2-5}}{x+3}$

42. $\lim_{x\to4} \frac{4-x}{5-\sqrt{x^2+9}}$

Limits with trigonometric functions Find the limits in Exercises 43–50.

43. $\lim_{x\to0} (2\sin x - 1)$

44. $\lim_{x\to0} \sin^2 x$

45. $\lim_{x\to0} \sec x$

46. $\lim_{x\to0} \tan x$

47. $\lim_{x\to0} \frac{1+x+\sin x}{3\cos x}$

48. $\lim_{x\to0} (x^2-1)(2-\cos x)$

49. $\lim_{x\to-\pi} \sqrt{x+4}\cos(x+\pi)$

50. $\lim_{x\to0} \sqrt{7+\sec^2 x}$

Using Limit Rules

51. Suppose $\lim_{x\to0} f(x) = 1$ and $\lim_{x\to0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to0} \frac{2f(x)-g(x)}{(f(x)+7)^{2/3}} = \frac{\lim_{x\to0}(2f(x)-g(x))}{\lim_{x\to0}(f(x)+7)^{2/3}} \quad \text{(a)}$$

$$= \frac{\lim_{x\to0} 2f(x) - \lim_{x\to0} g(x)}{\left(\lim_{x\to0}(f(x)+7)\right)^{2/3}} \quad \text{(b)}$$

$$= \frac{2\lim_{x\to0} f(x) - \lim_{x\to0} g(x)}{\left(\lim_{x\to0} f(x) + \lim_{x\to0} 7\right)^{2/3}} \quad \text{(c)}$$

$$= \frac{(2)(1)-(-5)}{(1+7)^{2/3}} = \frac{7}{4}$$

52. Let $\lim_{x\to1} h(x) = 5$, $\lim_{x\to1} p(x) = 1$, and $\lim_{x\to1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to1} \frac{\sqrt{5h(x)}}{p(x)(4-r(x))} = \frac{\lim_{x\to1}\sqrt{5h(x)}}{\lim_{x\to1}(p(x)(4-r(x)))} \quad \text{(a)}$$

$$= \frac{\sqrt{\lim_{x\to1} 5h(x)}}{\left(\lim_{x\to1} p(x)\right)\left(\lim_{x\to1}(4-r(x))\right)} \quad \text{(b)}$$

$$= \frac{\sqrt{5\lim_{x\to1} h(x)}}{\left(\lim_{x\to1} p(x)\right)\left(\lim_{x\to1} 4 - \lim_{x\to1} r(x)\right)} \quad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4-2)} = \frac{5}{2}$$

53. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

a. $\lim_{x\to c} f(x)g(x)$ **b.** $\lim_{x\to c} 2f(x)g(x)$

c. $\lim_{x\to c} (f(x)+3g(x))$ **d.** $\lim_{x\to c} \frac{f(x)}{f(x)-g(x)}$

54. Suppose $\lim_{x\to4} f(x) = 0$ and $\lim_{x\to4} g(x) = -3$. Find

a. $\lim_{x\to4} (g(x)+3)$ **b.** $\lim_{x\to4} xf(x)$

c. $\lim_{x\to4} (g(x))^2$ **d.** $\lim_{x\to4} \frac{g(x)}{f(x)-1}$

55. Suppose $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$. Find

a. $\lim_{x\to b} (f(x)+g(x))$ **b.** $\lim_{x\to b} f(x)\cdot g(x)$

c. $\lim_{x\to b} 4g(x)$ **d.** $\lim_{x\to b} f(x)/g(x)$

56. Suppose that $\lim_{x\to-2} p(x) = 4$, $\lim_{x\to-2} r(x) = 0$, and $\lim_{x\to-2} s(x) = -3$. Find

a. $\lim_{x\to-2} (p(x)+r(x)+s(x))$

b. $\lim_{x\to-2} p(x)\cdot r(x)\cdot s(x)$

c. $\lim_{x\to-2} (-4p(x)+5r(x))/s(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to0} \frac{f(x+h)-f(x)}{h}$$

occur frequently in calculus. In Exercises 57–62, evaluate this limit for the given value of x and function f.

57. $f(x) = x^2, \quad x = 1$

58. $f(x) = x^2, \quad x = -2$

59. $f(x) = 3x - 4, \quad x = 2$

60. $f(x) = 1/x, \quad x = -2$

61. $f(x) = \sqrt{x}, \quad x = 7$

62. $f(x) = \sqrt{3x+1}, \quad x = 0$

Using the Sandwich Theorem

63. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to0} f(x)$.

64. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to0} g(x)$.

65. a. It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x\sin x}{2-2\cos x} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to0} \frac{x\sin x}{2-2\cos x}?$$

Give reasons for your answer.

T **b.** Graph $y = 1 - (x^2/6)$, $y = (x\sin x)/(2-2\cos x)$, and $y = 1$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

66. a. Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 9.9.) What, if anything, does this tell you about

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

T **b.** Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

Estimating Limits

T You will find a graphing calculator useful for Exercises 67–76.

67. Let $f(x) = (x^2 - 9)/(x + 3)$.

a. Make a table of the values of f at the points $x = -3.1$, $-3.01, -3.001$, and so on as far as your calculator can go. Then estimate $\lim_{x \to -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \ldots$ instead?

b. Support your conclusions in part (a) by graphing f near $c = -3$ and using Zoom and Trace to estimate y-values on the graph as $x \to -3$.

c. Find $\lim_{x \to -3} f(x)$ algebraically, as in Example 7.

68. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.

a. Make a table of the values of g at the points $x = 1.4, 1.41$, 1.414, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x \to \sqrt{2}} g(x)$.

b. Support your conclusion in part (a) by graphing g near $c = \sqrt{2}$ and using Zoom and Trace to estimate y-values on the graph as $x \to \sqrt{2}$.

c. Find $\lim_{x \to \sqrt{2}} g(x)$ algebraically.

69. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.

a. Make a table of the values of G at $x = -5.9, -5.99, -5.999$, and so on. Then estimate $\lim_{x \to -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01$, $-6.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing G and using Zoom and Trace to estimate y-values on the graph as $x \to -6$.

c. Find $\lim_{x \to -6} G(x)$ algebraically.

70. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.

a. Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x \to 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing h near $c = 3$ and using Zoom and Trace to estimate y-values on the graph as $x \to 3$.

c. Find $\lim_{x \to 3} h(x)$ algebraically.

71. Let $f(x) = (x^2 - 1)/(|x| - 1)$.

a. Make tables of the values of f at values of x that approach $c = -1$ from above and below. Then estimate $\lim_{x \to -1} f(x)$.

b. Support your conclusion in part (a) by graphing f near $c = -1$ and using Zoom and Trace to estimate y-values on the graph as $x \to -1$.

c. Find $\lim_{x \to -1} f(x)$ algebraically.

72. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

a. Make tables of values of F at values of x that approach $c = -2$ from above and below. Then estimate $\lim_{x \to -2} F(x)$.

b. Support your conclusion in part (a) by graphing F near $c = -2$ and using Zoom and Trace to estimate y-values on the graph as $x \to -2$.

c. Find $\lim_{x \to -2} F(x)$ algebraically.

73. Let $g(\theta) = (\sin \theta)/\theta$.

a. Make a table of the values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta \to 0} g(\theta)$.

b. Support your conclusion in part (a) by graphing g near $\theta_0 = 0$.

74. Let $G(t) = (1 - \cos t)/t^2$.

a. Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t \to 0} G(t)$.

b. Support your conclusion in part (a) by graphing G near $t_0 = 0$.

75. Let $f(x) = x^{1/(1-x)}$.

a. Make tables of values of f at values of x that approach $c = 1$ from above and below. Does f appear to have a limit as $x \to 1$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $c = 1$.

76. Let $f(x) = (3^x - 1)/x$.

a. Make tables of values of f at values of x that approach $c = 0$ from above and below. Does f appear to have a limit as $x \to 0$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $c = 0$.

Theory and Examples

77. If $x^4 \le f(x) \le x^2$ for x in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x \to c} f(x)$? What can you say about the value of the limit at these points?

78. Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = -5.$$

Can we conclude anything about the values of f, g, and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x \to 2} f(x) = 0$? Give reasons for your answers.

79. If $\displaystyle\lim_{x \to 4} \frac{f(x) - 5}{x - 2} = 1$, find $\displaystyle\lim_{x \to 4} f(x)$.

80. If $\displaystyle\lim_{x \to -2} \frac{f(x)}{x^2} = 1$, find

a. $\displaystyle\lim_{x \to -2} f(x)$ **b.** $\displaystyle\lim_{x \to -2} \frac{f(x)}{x}$

81. a. If $\displaystyle\lim_{x \to 2} \frac{f(x) - 5}{x - 2} = 3$, find $\displaystyle\lim_{x \to 2} f(x)$.

b. If $\displaystyle\lim_{x \to 2} \frac{f(x) - 5}{x - 2} = 4$, find $\displaystyle\lim_{x \to 2} f(x)$.

82. If $\lim_{x\to 0} \frac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to 0} f(x)$ **b.** $\lim_{x\to 0} \frac{f(x)}{x}$

T **83.** **a.** Graph $g(x) = x\sin(1/x)$ to estimate $\lim_{x\to 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T **84.** **a.** Graph $h(x) = x^2\cos(1/x^3)$ to estimate $\lim_{x\to 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

COMPUTER EXPLORATIONS

Graphical Estimates of Limits

In Exercises 85–90, use a CAS to perform the following steps:

a. Plot the function near the point c being approached.

b. From your plot guess the value of the limit.

85. $\lim_{x\to 2} \frac{x^4 - 16}{x - 2}$

86. $\lim_{x\to -1} \frac{x^3 - x^2 - 5x - 3}{(x + 1)^2}$

87. $\lim_{x\to 0} \frac{\sqrt[3]{1 + x} - 1}{x}$

88. $\lim_{x\to 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$

89. $\lim_{x\to 0} \frac{1 - \cos x}{x\sin x}$

90. $\lim_{x\to 0} \frac{2x^2}{3 - 3\cos x}$

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. The early history of calculus saw controversy about the validity of the basic concepts underlying the theory. Apparent contradictions were argued over by both mathematicians and philosophers. These controversies were resolved by the precise definition, which allows us to replace vague phrases like "gets arbitrarily close to" in the informal definition with specific conditions that can be applied to any particular example. With a rigorous definition, we can avoid misunderstandings, prove the limit properties given in the preceding section, and establish many important limits.

To show that the limit of $f(x)$ as $x \to c$ equals the number L, we need to show that the gap between $f(x)$ and L can be made "as small as we choose" if x is kept "close enough" to c. Let us see what this requires if we specify the size of the gap between $f(x)$ and L.

EXAMPLE 1 Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it seems clear that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1) = 7$. However, how close to $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: what values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$|2x - 8| < 2$$
$$-2 < 2x - 8 < 2 \quad \text{Removing absolute value gives two inequalities.}$$
$$6 < 2x < 10 \quad \text{Add 8 to each term.}$$
$$3 < x < 5 \quad \text{Solve for } x.$$
$$-1 < x - 4 < 1. \quad \text{Solve for } x - 4.$$

Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Figure 2.15). ■

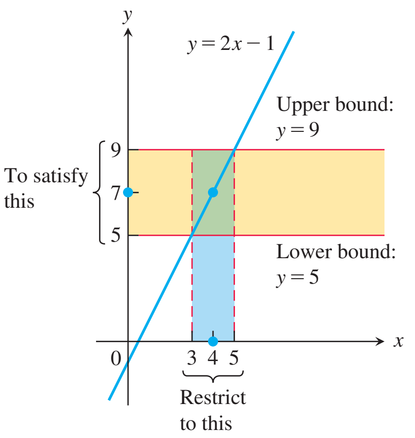


FIGURE 2.15 Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Example 1).

In the previous example we determined how close x must be to a particular value c to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a limit value L. To show that the limit of $f(x)$ as $x \to c$ actually equals L, we must be able to show that the gap between $f(x)$ and L can be made less than *any prescribed error*, no matter how